Relationship between Power Spectrum and Autocorrelation Function

Young W Lim

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Definition

if the second order density function
does not change with a shift in time origin

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any time t_1, t_2 and any real number Δ
if $X(t)$ is to be a **second-order stationary**

Auto-correlation function

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Definition

a **deterministic** Fourier transform $X(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

a **deterministic** signal $x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Definition

a **random** Fourier transform $X_T(\omega)$

$$X_T(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t} dt$$

a **random** signal $X_T(t)$

$$X_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_T(\omega) e^{j\omega t} d\omega$$

Definition

a **deterministic** sample signal $x(t)$

$$x(t) \Longleftrightarrow X(\omega)$$

a **random process** signal $X_T(t)$

$$X_T(t) \Longleftrightarrow X_T(\omega)$$

Definition

an estimate of the **mean function** of a **ramdom process** signal $X(t)$
the **sample average** of N sample signals $X_i(t)$

$$\hat{\hat{m}}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(t)$$

the **time average** of a **deterministic** sample signal $x(t)$

$$\overline{x}_T = A_T\left[x(t))\right] = \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

the **time autocorrelation** of a **deterministic** sample signal $x(t)$

$$R_T(\tau) = A_T\left[x(t)x(t+\tau)\right] = \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

Definition

For a **ramdom process** signal $X(t)$

$x_T(t)$ is the portion of a **sample function** $x(t)$ over the interval of $(-T, T)$

the **energy** over the interval $(-T, T)$

$$E(T) = \int_{-T}^{T} x_T^2(t)dt = \int_{-T}^{T} x^2(t)dt$$

the **average power** over the interval $(-T, T)$

$$P(T) = \frac{1}{2T}\int_{-T}^{T} x_T^2(t)dt = \frac{1}{2T}\int_{-T}^{T} x^2(t)dt$$

the **average power** of a **random process** signal $X(t)$

$$P_{XX} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E\left[X^2(t)\right]dt = A\left[E\left[X^2(t)\right]\right]$$

Definition

the **average power** P_{XX} of a **random process** signal $X(t)$
is given by the **time average** of its **second moment**

for a wide-sense stationary (**WSS**) process $X(t)$
the **average power** P_{XX} becomes a constant

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\left[X^2(t)\right] dt = A\left[E\left[X^2(t)\right]\right] = E\left[X^2(t)\right]$$

Definition

a **deterministic sample** signal $x_T(t)$

$$x_T(t) \Longleftrightarrow X_T(\omega)$$

Parseval's theorem

- a **deterministic** signal

$$\int_{-T}^{+T} |x_T(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

- a **random** signal

$$\int_{-T}^{+T} E\left[|x_T(t)|^2\right] dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E\left[|X_T(\omega)|^2\right] d\omega$$

Definition

For a **ramdom process** signal $X(t)$
$x_T(t)$ is the portion of a **sample function** $x(t)$ over the interval of $(-T, T)$
the **energy** over the interval $(-T, T)$

$$E(T) = \int_{-T}^{T} x_T^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** over the interval $(-T, T)$

$$P(T) = \frac{1}{2T}\int_{-T}^{T} x_T^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

the **average power** of a **random process** signal $X(t)$

$$P_{XX} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E\left[X^2(t)\right] dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lim_{T\to\infty}\frac{E\left[|X_T(\omega)|^2\right]}{2T} d\omega$$

Definition

the **power density spectrum** for the random process

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

the **power formula**

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega$$

Definition

To prove

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{+j\omega t}d\omega = A[R_{XX}(t,t+\tau)]}$$

Start with

$$\boxed{S_{XX}(\omega) = \lim_{T\to\infty}\frac{E\left[|X_T(\omega)|^2\right]}{2T}}$$

Definition

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

$$\begin{aligned} S_{XX}(\omega) &= \lim_{T\to\infty} \frac{1}{2T} E\left[X_T^*(\omega) X_T(\omega)\right] \\ &= E\left[\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{+T} X(t_1) e^{+j\omega t_1}\,dt_1 \int_{-T}^{+T} X(t_2) e^{-j\omega t_2}\,dt_2\right\}\right] \\ &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} E\left[X(t_1)X(t_2)\right] e^{-j\omega(t_2-t_1)}\,dt_2\,dt_1 \end{aligned}$$

$$X_T(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t}\,dt = \int_{-T}^{T} X(t) e^{-j\omega t}\,dt$$

Definition

$$\boxed{E\left[X(t_1)X(t_2)\right] = R_{XX}(t_1, t_2)}$$

$$\begin{aligned} S_{XX}(\omega) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} E\left[X(t_1)X(t_2)\right] e^{-j\omega(t_2 - t_1)} dt_2 dt_1 \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_2 dt_1 \end{aligned}$$

Definition

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_2 dt_1$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_2 dt_1 \right\} e^{+j\omega\tau} d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega(\tau - t_1 - t_2)} d\omega \right\} dt_2 dt_1$$

Definition

$$\int_{-\infty}^{+\infty} e^{+j\omega(\tau - t_1 - t_2)} d\omega = 2\pi\delta(t_1 - t_2 - \tau)$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega t} d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} R_{XX}(t_1, t_2) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega(\tau - t_1 - t_2)} d\omega \right\} dt_2 dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} \{R_{XX}(t_1, t_2)\delta(t_1 - t_2 - \tau)\} dt_2 dt_1$$

Definition

$$\boxed{2\pi\delta(\tau - t_1 + t_2) = 2\pi\delta(t_1 - t_2 - \tau)}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{+j\omega t}d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T}\{R_{XX}(t_2,t_1)\delta(t_1 - t_2 - \tau)\}\,dt_2 dt_1$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} R_{XX}(t, t+\tau)dt$$

$$-T < t + \tau < +T$$

Definition

$$\boxed{A[R_{XX}(t,t+\tau)]} = \lim_{T\to\infty} \int_{-T}^{+T} R_{XX}(t,t+\tau)dt$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{+j\omega t}d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} R_{XX}(t,t+\tau)dt$$

$$= \boxed{A[R_{XX}(t,t+\tau)]}$$

$$-T < t+\tau < +T$$

Definition

$$A[R_{XX}(t, t+\tau)] \Longleftrightarrow S_{XX}(\omega)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{+j\omega t} d\omega = A[R_{XX}(t, t+\tau)]$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t+\tau)] e^{-j\omega\tau} d\tau$$

Definition

$$R_{XX}(\tau) = A[R_{XX}(t, t+\tau)] = \lim_{T\to\infty} \int_{-T}^{+T} R_{XX}(t, t+\tau)dt$$

$$A[R_{XX}(t, t+\tau)] \Longleftrightarrow S_{XX}(\omega)$$

$$R_{XX}(\tau) \Longleftrightarrow S_{XX}(\omega)$$

Definition

$$R_{XX}(\tau) \Longleftrightarrow S_{XX}(\omega)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{+j\omega t}d\omega = R_{XX}(\tau)$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau$$

Definition

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{+j\omega t}d\omega = A[R_{XX}(t,t+\tau)]$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t,t+\tau)]e^{-j\omega\tau}d\tau$$

$$A[R_{XX}(t,t+\tau)] \Longleftrightarrow S_{XX}(\omega)$$

WSS $X(t)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{+j\omega t}d\omega = R_{XX}(\tau)$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau$$

$$R_{XX}(\tau) \Longleftrightarrow S_{XX}(\omega)$$