

Relationship between Power Spectrum and Autocorrelation Function

Young W Lim

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles, Jr. and B. Shi

Second-Order Stationary Process

N Gaussian random variables

Definition

if the second order density function
does not change with a shift in time origin

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any time t_1, t_2 and any real number Δ
if $X(t)$ is to be a **second-order stationary**

Auto-correlation function

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$$

Fourier transform - deterministic $x(t)$

N Gaussian random variables

Definition

a **deterministic** Fourier transform $X(\omega)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

a **deterministic** signal $x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier transform - random $X_T(t)$

N Gaussian random variables

Definition

a **random** Fourier transform $X_T(\omega)$

$$X_T(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t} dt$$

a **random** signal $X_T(t)$

$$X_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_T(\omega) e^{j\omega t} d\omega$$

Fourier transform - $x(t)$ and $X_T(t)$

N Gaussian random variables

Definition

a **deterministic** sample signal $x(t)$

$$x(t) \iff X(\omega)$$

a **random process** signal $X_T(t)$

$$X_T(t) \iff X_T(\omega)$$

Time average and ergodicity

N Gaussian random variables

Definition

an estimate of the **mean function** of a **random process** signal $X(t)$
the **sample average** of N sample signals $X_i(t)$

$$\hat{m}_X(t) = \frac{1}{N} \sum_{i=1}^N X_i(t)$$

the **time average** of a **deterministic** sample signal $x(t)$

$$\bar{x}_T = A_T[x(t)] = \frac{1}{2T} \int_{-T}^T x(t) dt$$

the **time autocorrelation** of a **deterministic** sample signal $x(t)$

$$R_T(\tau) = A_T[x(t)x(t+\tau)] = \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt$$

Energy and Average Power (I) time domain

N Gaussian random variables

Definition

For a **random process** signal $X(t)$

$x_T(t)$ is the portion of a **sample function** $x(t)$ over the interval of $(-T, T)$
the **energy** over the interval $(-T, T)$

$$E(T) = \int_{-T}^T x_T^2(t) dt = \int_{-T}^T x^2(t) dt$$

the **average power** over the interval $(-T, T)$

$$P(T) = \frac{1}{2T} \int_{-T}^T x_T^2(t) dt = \frac{1}{2T} \int_{-T}^T x^2(t) dt$$

the **average power** of a **random process** signal $X(t)$

$$P_{XX} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[X^2(t)] dt = A[E[X^2(t)]]$$

Average Power P_{XX}

N Gaussian random variables

Definition

the **average power** P_{XX} of a **random process** signal $X(t)$ is given by the **time average** of its **second moment**

for a wide-sense stationary (**WSS**) process $X(t)$ the **average power** P_{XX} becomes a constant

$$P_{XX} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[X^2(t)] dt = A[E[X^2(t)]] = E[X^2(t)]$$

Definition

a **deterministic** sample signal $x_T(t)$

$$x_T(t) \iff X_T(\omega)$$

Parseval's theorem

- a **deterministic** signal

$$\int_{-T}^{+T} |x_T(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

- a **random** signal

$$\int_{-T}^{+T} E [|x_T(t)|^2] dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E [|X_T(\omega)|^2] d\omega$$

Energy and Average Power (II) frequency domain

N Gaussian random variables

Definition

For a **random process** signal $X(t)$

$x_T(t)$ is the portion of a **sample function** $x(t)$ over the interval of $(-T, T)$
the **energy** over the interval $(-T, T)$

$$E(T) = \int_{-T}^T x_T^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** over the interval $(-T, T)$

$$P(T) = \frac{1}{2T} \int_{-T}^T x_T^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

the **average power** of a **random process** signal $X(t)$

$$P_{XX} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[X^2(t)] dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T} d\omega$$

Power density spectrum and power formula

N Gaussian random variables

Definition

the **power density spectrum** for the random process

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

the **power formula**

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega$$

(1) Power Density Spectrum

N Gaussian random variables

Definition

To prove

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{+j\omega t} d\omega = A[R_{XX}(t, t + \tau)]$$

Start with

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

(2) Inverse Fourier Transform $X_T(t)$

N Gaussian random variables

Definition

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$\begin{aligned} S_{XX}(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{2T} E[X_T^*(\omega) X_T(\omega)] \\ &= E \left[\lim_{T \rightarrow \infty} \left\{ \frac{1}{2T} \int_{-T}^{+T} X(t_1) e^{+j\omega t_1} dt_1 \int_{-T}^{+T} X(t_2) e^{-j\omega t_2} dt_2 \right\} \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} E[X(t_1) X(t_2)] e^{-j\omega(t_2 - t_1)} dt_2 dt_1 \end{aligned}$$

$$X_T(\omega) = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t} dt = \int_{-T}^T X(t) e^{-j\omega t} dt$$

(3) AutoCorrelation $R_{XX}(t_1, t_2)$

N Gaussian random variables

Definition

$$E[X(t_1)X(t_2)] = R_{XX}(t_1, t_2)$$

$$\begin{aligned} S_{XX}(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} E[X(t_1)X(t_2)] e^{-j\omega(t_2-t_1)} dt_2 dt_1 \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2-t_1)} dt_2 dt_1 \end{aligned}$$

(4) Inverse Transform of $S_{XX}(\omega)$

N Gaussian random variables

Definition

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_2 dt_1$$

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_2 dt_1 \right\} e^{+j\omega \tau} d\omega \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega(\tau - t_1 - t_2)} d\omega \right\} dt_2 dt_1 \end{aligned}$$

(5) Impulse Function

N Gaussian random variables

Definition

$$\int_{-\infty}^{+\infty} e^{+j\omega(\tau-t_1-t_2)} d\omega = 2\pi\delta(t_1 - t_2 - \tau)$$

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega t} d\omega \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega(\tau-t_1-t_2)} d\omega \right\} dt_2 dt_1 \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} \{R_{XX}(t_1, t_2) \delta(t_1 - t_2 - \tau)\} dt_2 dt_1 \end{aligned}$$

(6) Impulse Function Property

N Gaussian random variables

Definition

$$2\pi\delta(\tau - t_1 + t_2) = 2\pi\delta(t_1 - t_2 - \tau)$$

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} \{R_{XX}(t_2, t_1) \delta(t_1 - t_2 - \tau)\} dt_2 dt_1 \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} R_{XX}(t, t + \tau) dt \\ & \quad -T < t + \tau < +T \end{aligned}$$

(7) Time Average of an Auto-Correlation

N Gaussian random variables

Definition

$$\boxed{A[R_{XX}(t, t + \tau)]} = \lim_{T \rightarrow \infty} \int_{-T}^{+T} R_{XX}(t, t + \tau) dt$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega t} d\omega$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} R_{XX}(t, t + \tau) dt$$

$$= \boxed{A[R_{XX}(t, t + \tau)]}$$

$$-T < t + \tau < +T$$

(8) Fourier Transform Pair

N Gaussian random variables

Definition

$$A[R_{XX}(t, t + \tau)] \iff S_{XX}(\omega)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{+j\omega t} d\omega = A[R_{XX}(t, t + \tau)]$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t + \tau)] e^{-j\omega\tau} d\tau$$

(9) a WSS process $X(t)$

N Gaussian random variables

Definition

$$R_{XX}(\tau) = A[R_{XX}(t, t + \tau)] = \lim_{T \rightarrow \infty} \int_{-T}^{+T} R_{XX}(t, t + \tau) dt$$

$$A[R_{XX}(t, t + \tau)] \iff S_{XX}(\omega)$$

$$R_{XX}(\tau) \iff S_{XX}(\omega)$$

(10) Time Average of an Auto-Correlation : a WSS case

N Gaussian random variables

Definition

$$R_{XX}(\tau) \iff S_{XX}(\omega)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega = R_{XX}(\tau)$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

Fourier Transform Pairs

N Gaussian random variables

Definition

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{+j\omega t} d\omega = A[R_{XX}(t, t + \tau)]$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t + \tau)] e^{-j\omega\tau} d\tau$$

$$A[R_{XX}(t, t + \tau)] \iff S_{XX}(\omega)$$

WSS $X(t)$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{+j\omega t} d\omega = R_{XX}(\tau)$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) \iff S_{XX}(\omega)$$

