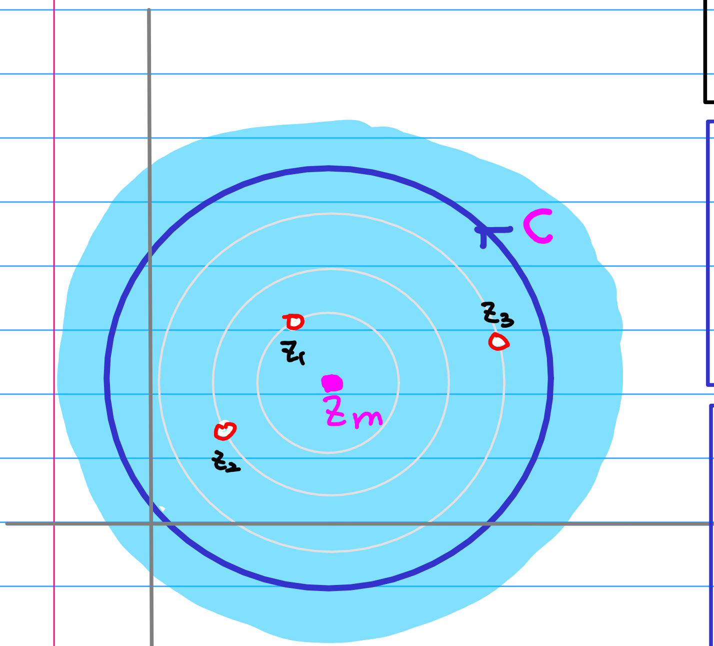
Laurent Series with Applications

20170327

Series Expansion at z_m



$$f(z) = \sum_{n=n_1}^{\infty} \boxed{a_n^{\{m\}}} (z - z_m)^n$$

$$\boxed{a_n^{\{m\}}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} \, dz$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

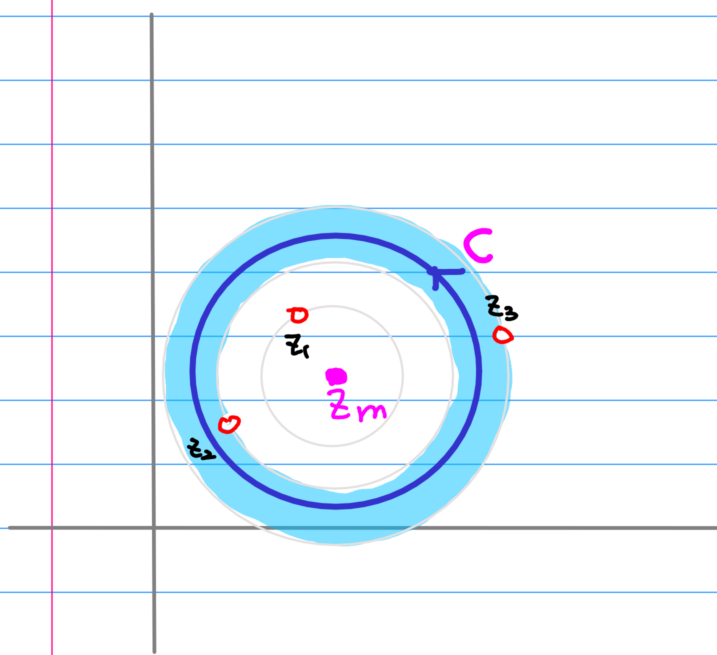
$n=-1$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i} \oint_C f(z) \, dz$$

$$= \sum_k \mathrm{Res}\left(f(z), z_k\right)$$

$$a_{-1}^{\{m\}} \neq \mathrm{Res}(f(z), z_m)$$

[Annular Region] ⇒ Laurent Series



* Even if z_m is a non-singular point of $f(z)$, z_m becomes a pole in the residue computation.

$$\boxed{\frac{f(z)}{(z-z_m)^{n+1}}} \quad \text{if } n \geq 0$$

$$\boxed{a_n^{\{m\}}} = \sum_k \operatorname{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

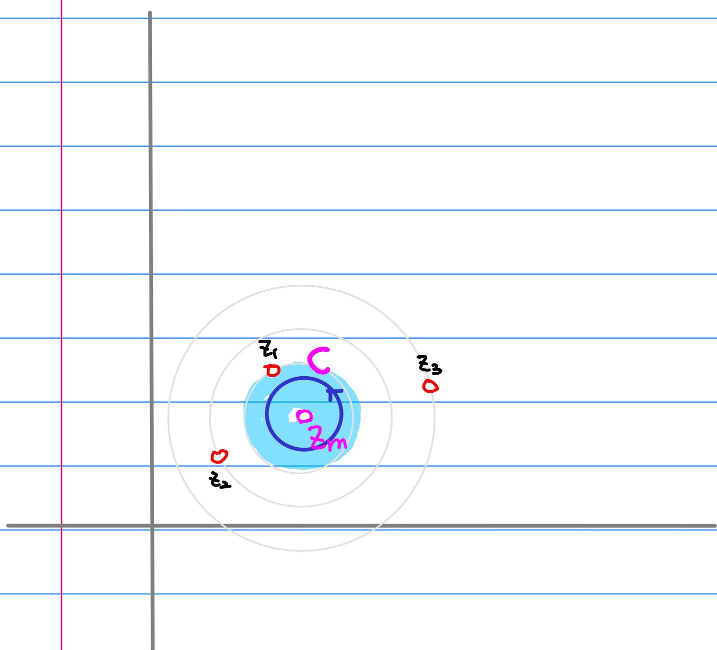
$$a_{-1}^{\{m\}} \neq \operatorname{Res}(f(z), z_m)$$

∵ residue is defined on a punctured open disk.

[Annular Region] & [z_m : isolated singularity]

a punctured open disk $\Rightarrow$ Residue,
Laurent Series

◎ only one pole is enclosed by C



$$f(z) = \sum_{n=n_1}^{\infty} \boxed{a_n^{\{m\}}} (z - z_m)^n$$

$$\boxed{a_n^{\{m\}}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}}\, dz$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k\right)$$

$n=$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

$$= \sum_k \mathrm{Res}\left(f(z), z_k\right)$$

$$\sum_k z_k = z_m$$

the only pole enclosed by C, is z_m

a punctured open disk

$$a_{-1}^{\{m\}} = \mathrm{Res}(f(z), z_m)$$

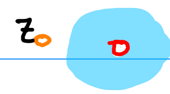
Isolated Singularity

$z = z_0$

- a **singularity** of a complex function f

- an **isolated singularity** of a complex function f

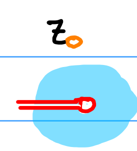
  if there exists some deleted neighborhood
  or punctured open disk of z_0
  where $f(z)$ is analytic
  $0 < |z - z_0| < R$



- a non-isolated singularity

  if every neighborhood of z_0 contains
  at least one singularity of f other than z_0

  the branch point $z=0$ $\quad$ $\operatorname{Ln} z$



  every neighborhood of $z=0$
  contains points on the negative real axis

branch cut : non-positive real axis

$\mathrm{Ln}\, z$

Principal Argument $\quad \mathrm{Arg}(z) = \theta \quad -\pi < \theta \le \pi$

$z \neq 0$ & $\theta = \arg z$

$$\boxed{\ln z = \log_e |z| + i(\theta + 2n\pi)} \quad n = 0, \pm 1, \pm 2, \cdots$$

Principal Value

$$\boxed{\mathrm{Ln}\, z = \log_e |z| + i\, \mathrm{Arg}\, z}$$

$\mathrm{Arg}\, z$: unique $\longrightarrow$ $\mathrm{Ln}\, z$: unique for $z \neq 0$

$f(z) = \mathrm{Ln}\, z$ not continuous at $z = 0$

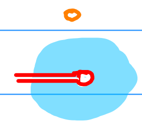
$f(0)$ not defined $\longleftarrow$ $\log_e 0$ not defined

$f(z) = \mathrm{Ln}\, z$ discontinuous at the negative real axis

$\longleftarrow$ $\mathrm{Arg}\, z$ discontinuous

for $x_0 < 0$ $\quad \lim\limits_{z \to z_0} \mathrm{Arg}\, z = \pm \pi$

the non-positive real axis : the branch cut

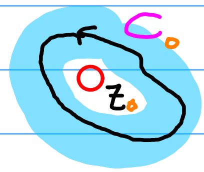


A punctured open disk

if C encloses only one pole z_0,

and the expansion at that pole z_0 is assumed,

then



$$\boxed{a^{\{0\}}_{-1}} = \frac{1}{2\pi i} \oint_{C_0} f(z)\, dz = \mathbf{Res}(f(z), z_0)$$

Let $$\boxed{\tilde{a}^{\{m\}}_{-1} = \mathbf{Res}(f(z), z_m)}$$ notation $\sim$

the residue of $f(z)$ at z_m

using C_m which is in the punctured open disk ROC

$$\boxed{f(z) = \sum_{n=-\infty}^{+\infty} a^{\{m\}}_n (z - z_m)^n}$$

Residues Computation

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s)\, ds \Rightarrow \oint_C f(s)\, ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s)\, ds = \operatorname{Res}(f(z), z_0)$$

$$= \begin{cases} \lim\limits_{z \to z_0} (z - z_0) f(z) & \text{(simple)} \\ \dfrac{1}{(n-1)!} \lim\limits_{z \to z_0} \dfrac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & \text{(order } n\text{)} \end{cases}$$

$$f(z) = \frac{\boxed{\quad\quad}}{(z - z_0)\boxed{\quad\quad}}$$

$$\Rightarrow \operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} \frac{\boxed{\quad\quad}}{\boxed{\quad\quad}}$$

$$f(z) = \frac{\boxed{\quad\quad}}{(z - z_0)^n\boxed{\quad\quad}}$$

$$\Rightarrow \operatorname{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left(\frac{d^{n-1}}{dz^{n-1}} \frac{\boxed{\quad\quad}}{\boxed{\quad\quad}} \right)$$

z_0 : expansion center

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

z_0 : simple pole $\qquad \dfrac{1}{z-z_0}$

z_0 : n-th order pole $\qquad \dfrac{1}{(z-z_0)^n}$

Ⅰ z_0 : expansion center & simple pole $(a_{-1} \neq 0)$

$$f(z) = \boxed{\frac{a_{-1}}{(z-z_0)}} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Ⅱ z_0 : expansion center & n-th order pole $(a_{-n} \neq 0)$

$$f(z) = \boxed{\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)}} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

(I) z_0 : expansion center & simple pole $(a_{-1} \neq 0)$

$$f(z) = \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$(z-z_0)f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 + \cdots$$

$$\boxed{\lim_{z \to z_0} (z-z_0) f(z) = a_{-1}}$$

(II) z_0 : expansion center & n-th order pole $(a_{-n} \neq 0)$

$\boxed{n=2}$

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + a_1(z-z_0)^3 + a_2(z-z_0)^4 + \cdots$$

$$\frac{d}{dz}(z-z_0)^2 f(z) = a_{-1} + 2a_0(z-z_0) + 3a_1(z-z_0)^2 + 4a_2(z-z_0)^3 + \cdots$$

$$\boxed{\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) = a_{-1}}$$

L'Hopital's Theorem

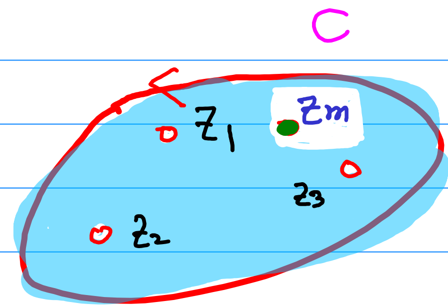
$$\lim_{z \to z_0} \frac{g(z)}{h(z)} = \frac{g'(z_0)}{h'(z_0)}$$

General Series Expansion

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right)$$



$\boxed{C}$ is in the same region of analyticity of $f(z)$

typically a circle centered on z_m non-annular ok

$\boxed{z_k}$ within C : singularities of $\boxed{\dfrac{f(z)}{(z-z_m)^{n+1}}}$

$\boxed{n_1 = n_{f,m}}$ depends on $\boxed{f(z)}$, $\boxed{z_m}$

$\boxed{a_n^{\{m\}}}$ depends on $\boxed{f(z)}$, $\boxed{z_m}$, $\boxed{\text{region of analyticity}}$

Whether $f(z)$ is singular at z_m or not

or singular at other points between $\boxed{z}$ and z_m

We can expand $f(z)$ about any point z_m

over powers of $(z - z_m)$.

Whether $f(z)$ is singular at z_m or not

or singular at other points between $\boxed{z}$ and z_m

We can expand $f(z)$ about any point z_m

over powers of $(z - z_m)$.

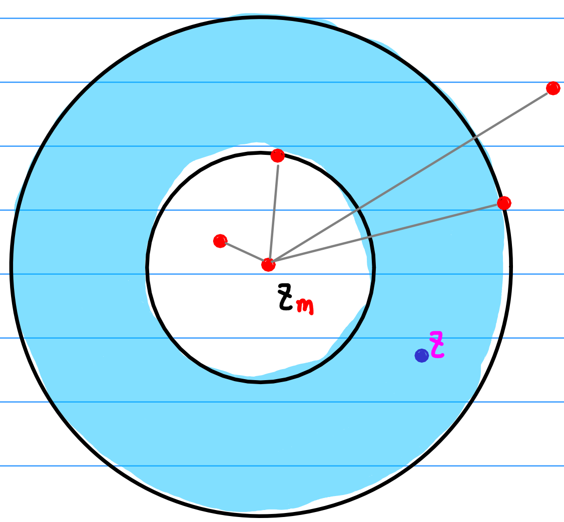
$\boxed{z}$ evaluation point

z_m expansion center

$\boxed{z_k}$ one of K poles of $\boxed{\dfrac{f(z)}{(z-z_m)^{n+1}}}$

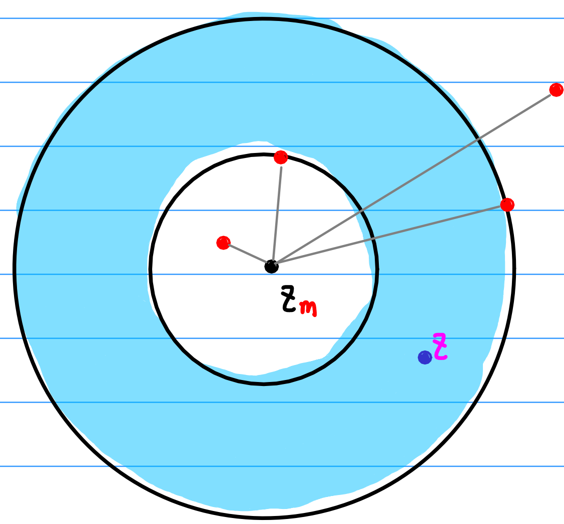
$= \{\text{poles of } f(z)\} \cup \{\text{pole } z_m \text{ depending on } n\}$

Laurent Series



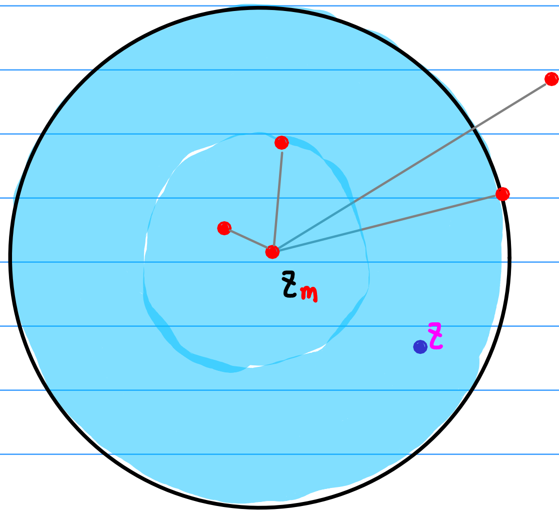
singluar z_m

Laurent Series



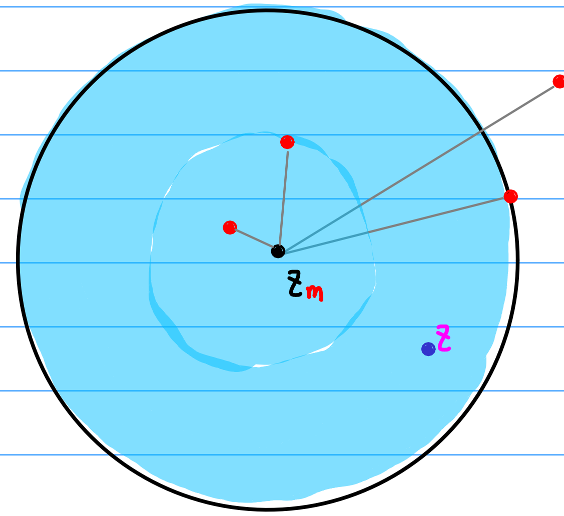
non-singluar z_m

Non-Laurent Series

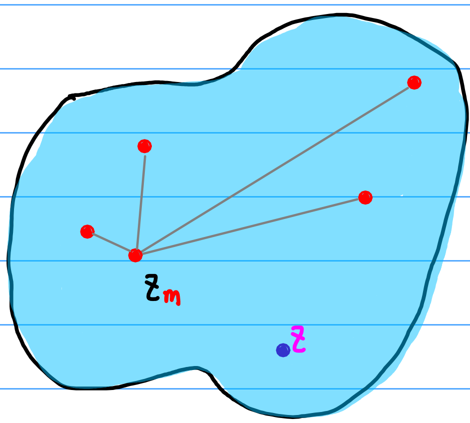


singluar z_m

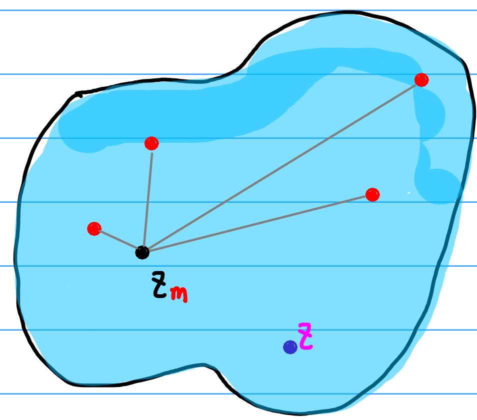
Non-Laurent Series



non-singluar z_m



singluar z_m



non-singluar z_m

Taylor Series

Non-Laurent Series

Taylor Series

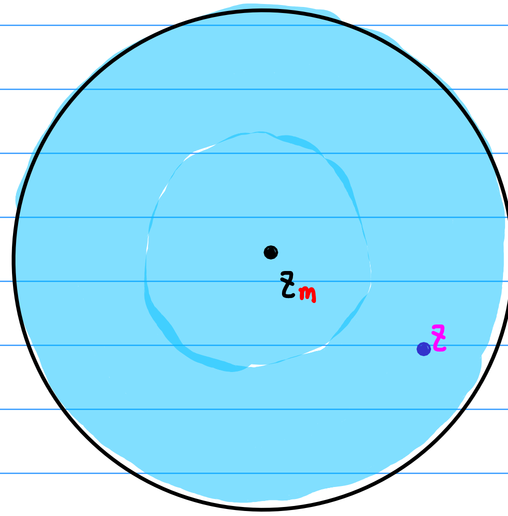
$f(z)$ analytic on and within C

→ no poles

→ z_m becomes the only pole of the residue of

$$\boxed{\frac{f(z)}{(z-z_m)^{n+1}}}$$

when $n \geq 0$



non-singluar z_m

no singular points on and within C

$n \geq 0$ →

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz$$

$$= \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_m\right)$$

$$= \frac{1}{n!} f^{(n)}(z_m)$$

analytic $f(z) \rightarrow$ Taylor Series

$\boxed{n_1 = n_{f,m}}$ depends on $\boxed{f(z)}$, $\boxed{z_m}$

$f(z)$ analytic on and within C

→ no poles

→ z_m becomes the only pole of the residue of $\boxed{\dfrac{f(z)}{(z-z_m)^{n+1}}}$ when $n \geq 0$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz$$

$$= \begin{cases} 0 & (n < 0) \\ \mathrm{Res}\left(\dfrac{f(z)}{(z-z_m)^{n+1}}, z_m\right) & (n \geq 0) \end{cases}$$

$$\operatorname{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_m\right) \quad (n \geq 0)$$

$n=0$
$$\operatorname{Res}\left(\frac{f(z)}{(z-z_m)^{1}}, z_m\right) = \lim_{z\to z_m} (z-z_m)\frac{f(z)}{(z-z_m)} = f(z_m)$$

$n=1$
$$\operatorname{Res}\left(\frac{f(z)}{(z-z_m)^{2}}, z_m\right) = \frac{1}{(2-1)} \lim_{z\to z_m} \frac{d}{dz}\left((z-z_m)^2\frac{f(z)}{(z-z_m)^2}\right) = f'(z_m)$$

$n=2$
$$\operatorname{Res}\left(\frac{f(z)}{(z-z_m)^{3}}, z_m\right) = \frac{1}{(3-1)!} \lim_{z\to z_m} \frac{d^2}{dz^2}\left((z-z_m)^3\frac{f(z)}{(z-z_m)^3}\right) = \frac{1}{2!} f''(z_m)$$

n
$$\operatorname{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_m\right) = \frac{1}{n!} \lim_{z\to z_m} \frac{d^n}{dz^n}\left((z-z_m)^{n+1}\frac{f(z)}{(z-z_m)^{n+1}}\right) = \frac{1}{n!} f^{(n)}(z_m)$$

$\operatorname{Res}(G(z), z_0)$

$$\begin{cases} \boxed{\lim_{z\to z_0} (z-z_0)\, G(z) = a_{-1}} & \text{simple pole } z_0 \\ \boxed{\frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n\, G(z) = a_{-1}} & n\text{-th order pole } z_0 \end{cases}$$

$f(z)$ analytic on and within C

→ no poles

→ z_m becomes the only pole when $n \geq 0$

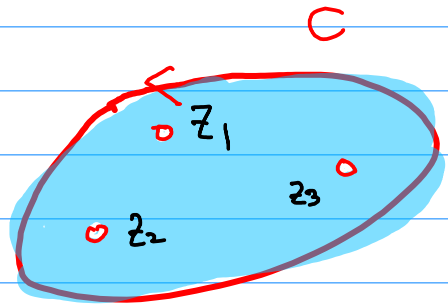
$$\boxed{\frac{f(z)}{(z-z_m)^{n+1}}}$$

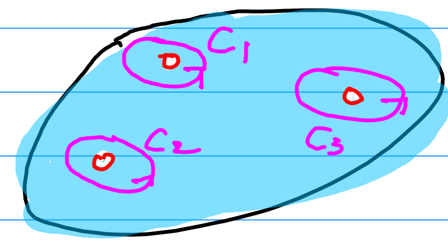
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz$$

$$= \begin{cases} 0 & (n<0) \\ \mathrm{Res}\left(\dfrac{f(z)}{(z-z_m)^{n+1}}, z_m\right) & (n \geq 0) \end{cases}$$

$$= \begin{cases} 0 & (n<0) \\ \dfrac{1}{n!} f^{(n)}(z_m) & (n \geq 0) \end{cases}$$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_m)}{k!} (z-z_m)^k$$ Taylor Series





$$\oint_C f(z)\,dz = 2\pi j \sum_{k=1}^{K} \tilde{a}_{-1}^{\{k\}} = 2\pi j \sum_{k=1}^{K} \mathrm{Res}(f(z), z_k)$$

residue theorem

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

general series

$$a_n^{\{m\}} = \sum_{k=1}^{K} \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}} , z_k \right)$$

general formula

C encloses K poles $(z_1, z_2, \dots, z_k, \dots, z_K)$ of $\frac{f(z)}{(z-z_m)^{n+1}}$

C_k encloses only the k-th pole (z_k)

$\tilde{a}_{-1}^{\{k\}}$ the **residue** of the k-th pole z_k enclosed by C_k

(Laurent series coefficients on a punctured open disk)

Laurent's Theorem

f : analytic within the $\boxed{\text{annular}}$ domain D

$$\boxed{r < |z - z_0| < R}$$

then

$$\boxed{f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k},$$

valid for $\boxed{r < |z - z_0| < R}$ (ROC)

The coefficients a_k are given by

$$\boxed{a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}}\, ds}, \qquad k = 0, \pm 1, \pm 2, \cdots$$

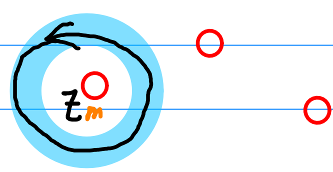
C : a simple closed curve
that lies entirely within D
that encloses z_0

Curve C & Domain D of the Laurent Series

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

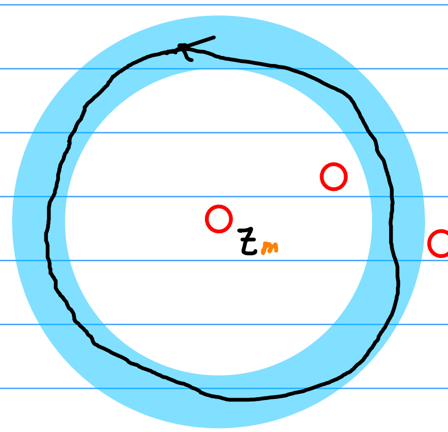
$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz' \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right) \end{aligned}}$$

a punctured open disk

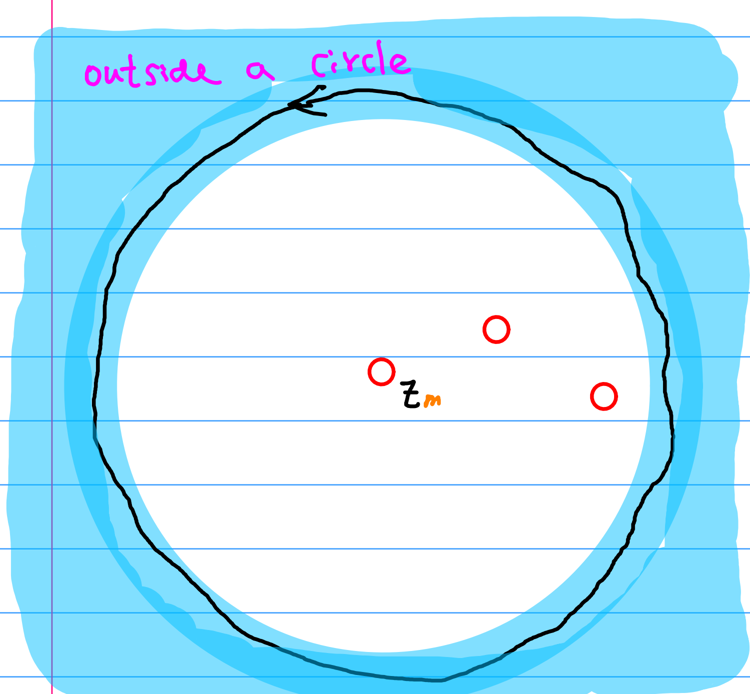


annular region

ring

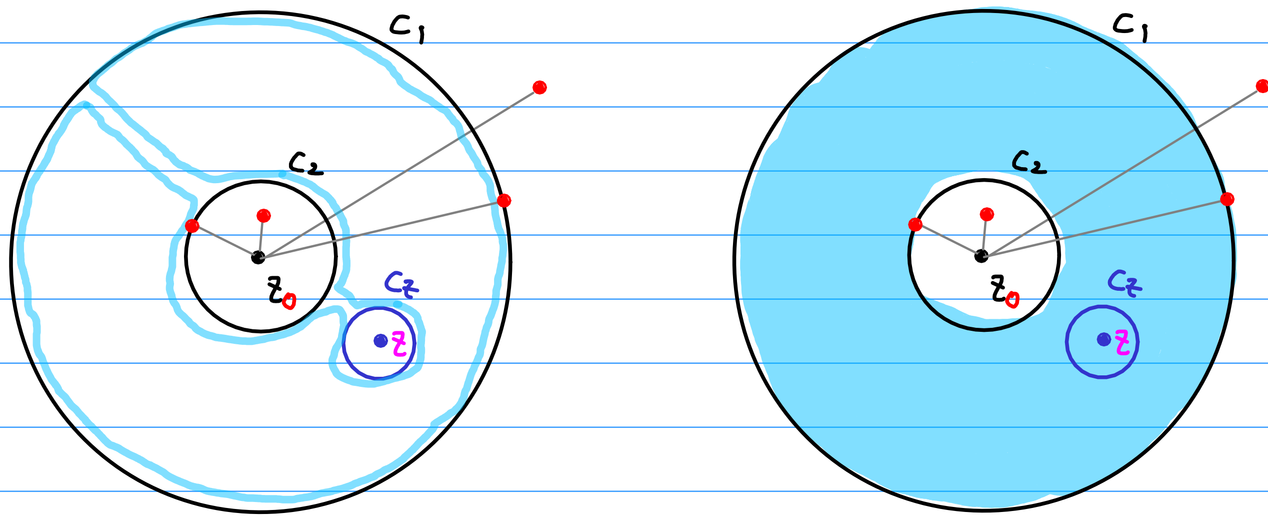


annular region



annular region

Expansion Points and Evaluation Points



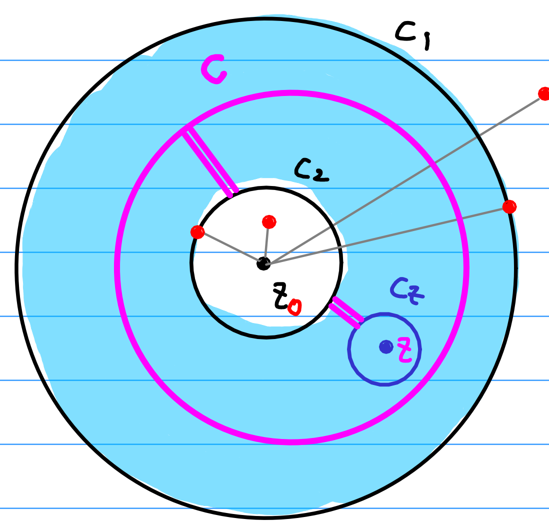
z_0 : expansion point

z : evaluation point (in ROC)

$\boxed{\dfrac{f(z)}{(z-z_0)}}$ is analytic between C_1 & C_2

deformation theorem $C_1 - C_2$ coincide

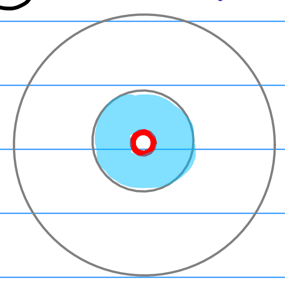
common contour C



3 types of annular region

all Laurent Series ROC

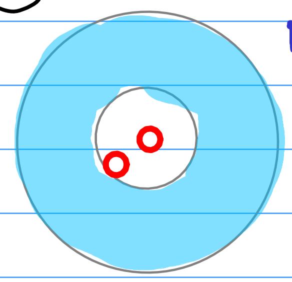
(A) only one pole



punctured open disk

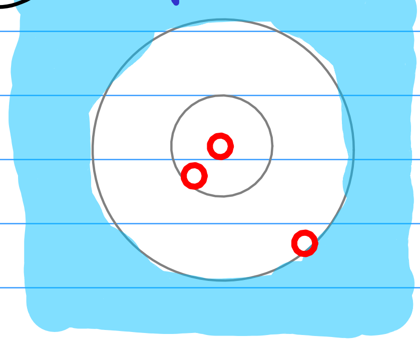
only this region defines a residue

(B) more than one pole



ring

(C) all poles



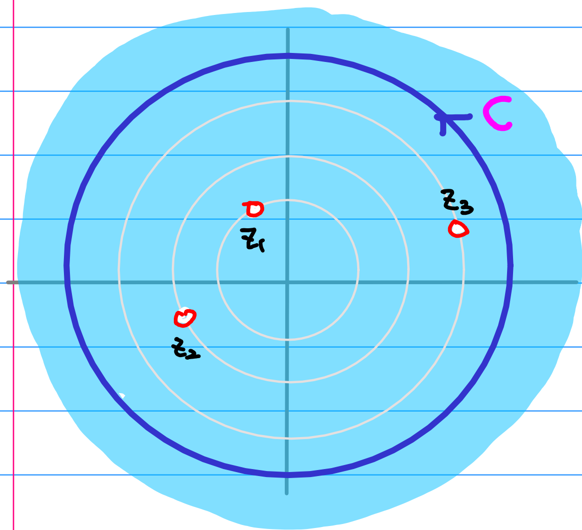
outside a circle

	annular region			non-annular region
	punctured	ring	outside circle	
singular center z_m	$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$			
non-singular center z_m	$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} dz' = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$			

	annular region			non-annular region
	punctured	ring	outside circle	
singular center z_m	Laurent Series			X
non-singular center z_m	Laurent Series			X

	annular region			non-annular region
	punctured	ring	utside circle	
singular center z_m	residue	x	x	x
non-singular center z_m	x	x	x	x

Series Expansion at $z=0$



$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} z^n}$$

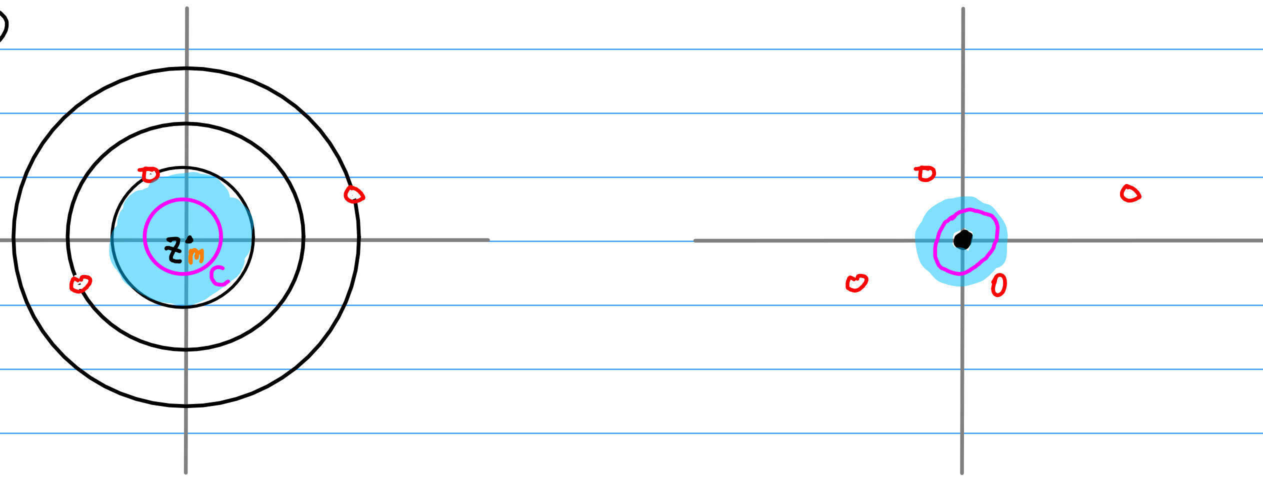
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\, dz$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

Poles z_k

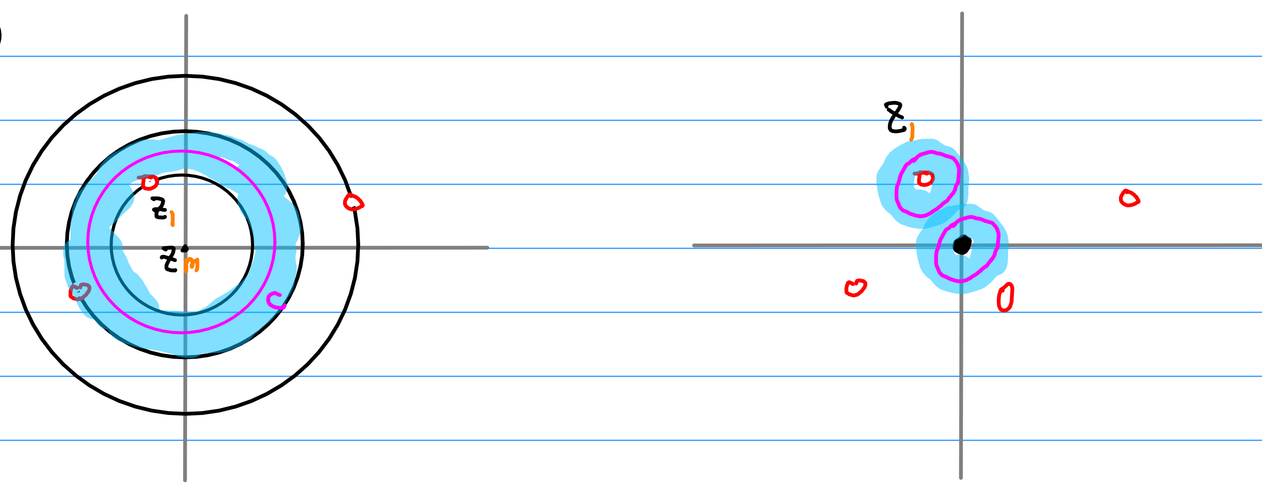
$n \geq 0$	$z_1, z_2, z_3, 0$
$n < 0$	z_1, z_2, z_3

①



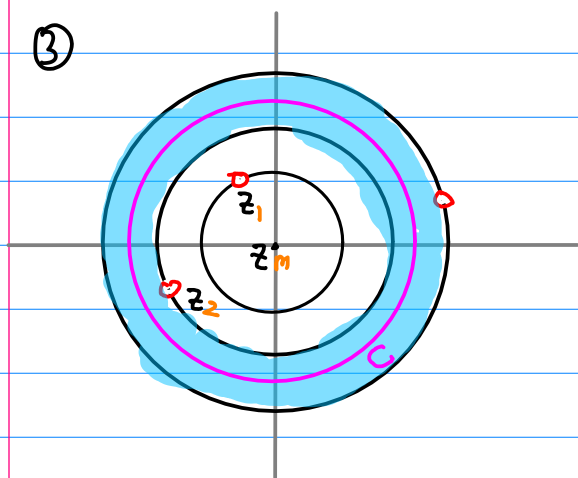
$$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) \quad (n \geq 0)$$

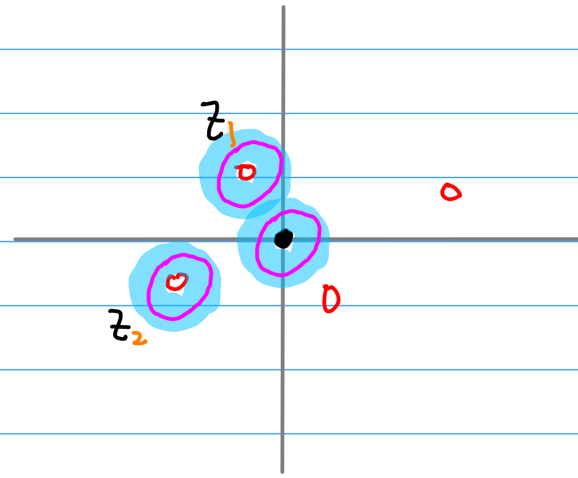
②



$$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) \quad (n \geq 0)$$

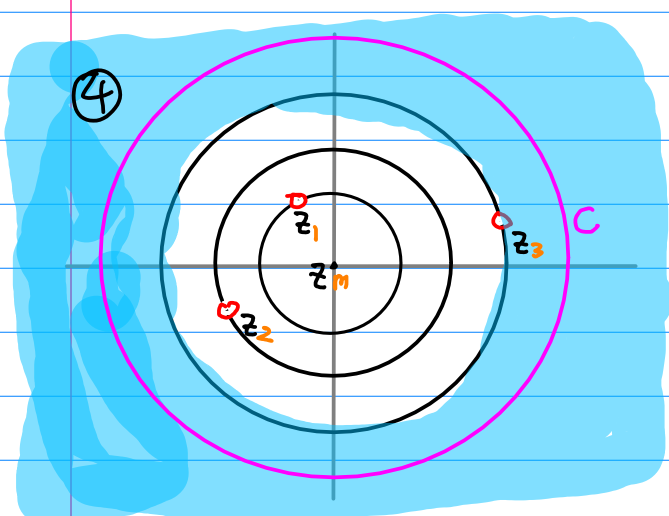
③

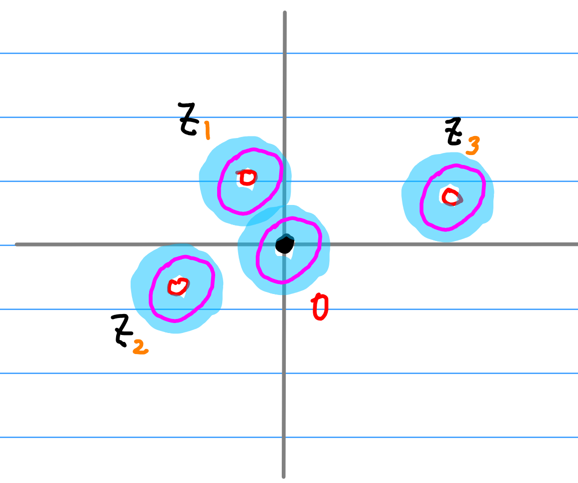




$$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_2\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) \quad (n \geq 0)$$

④

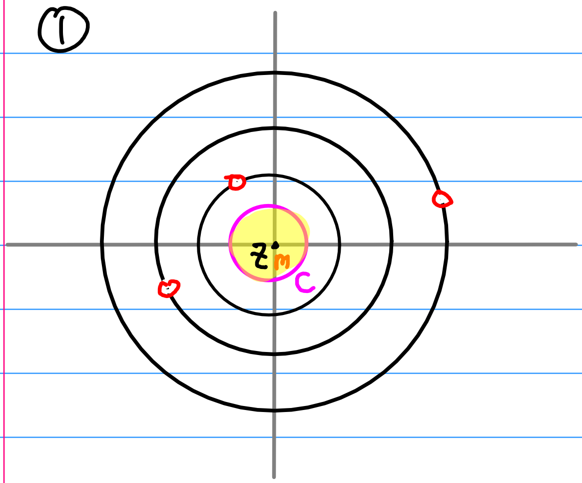


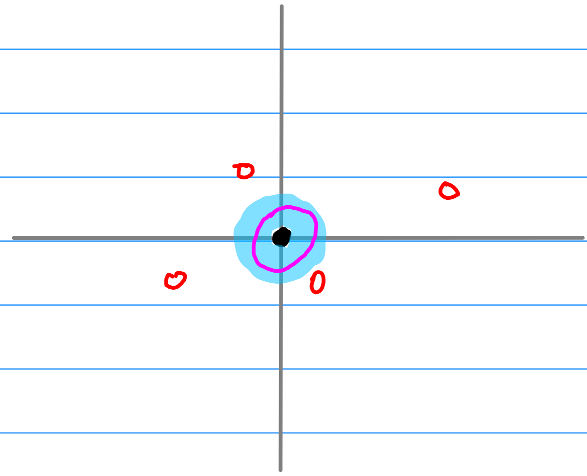


$$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_3\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_2\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$$

$(n \geq 0)$

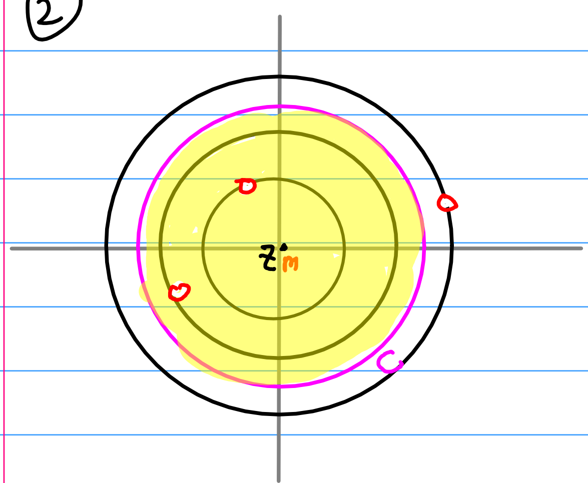
①

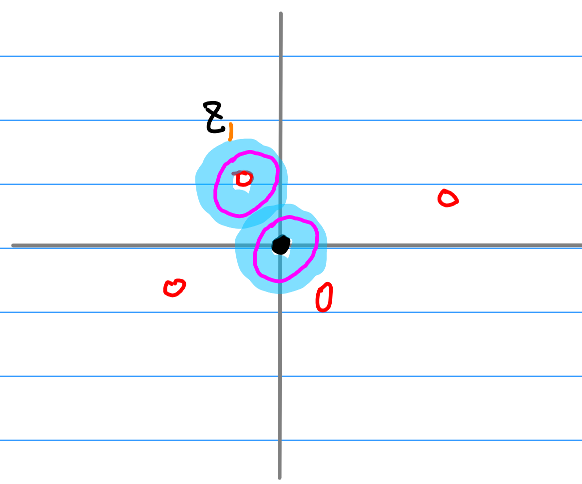




$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$ $(n \geq 0)$

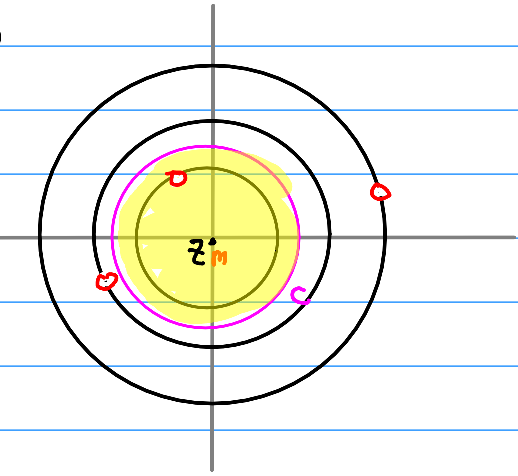
②

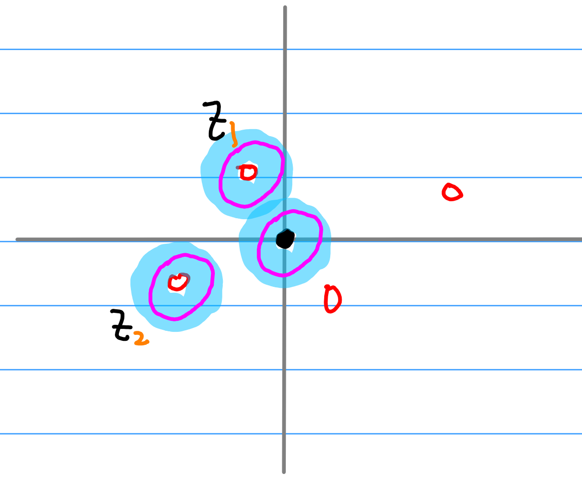




$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$ $(n \geq 0)$

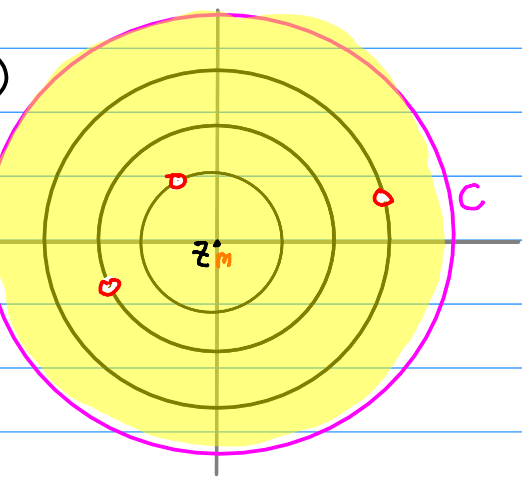
③

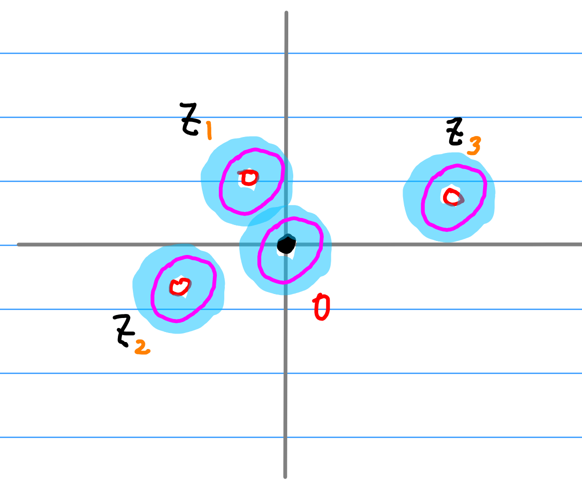




$$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_2\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right) \quad (n \geq 0)$$

④



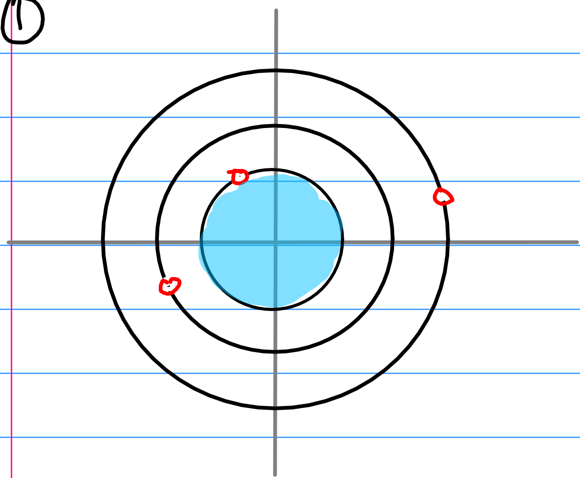


$$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_3\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_2\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_1\right) + \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$$

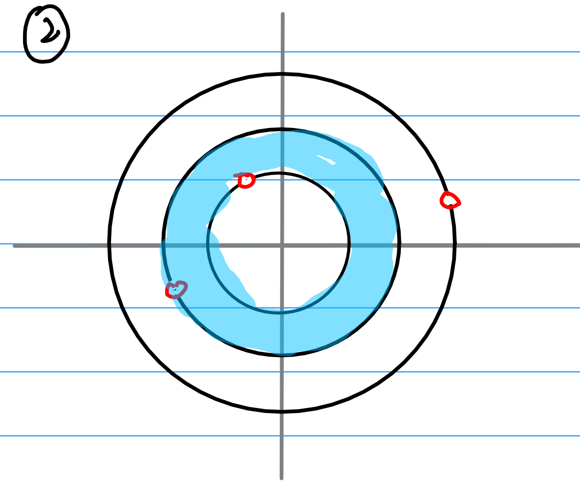
$(n \geq 0)$

Different D, Different Laurent Series

1



2



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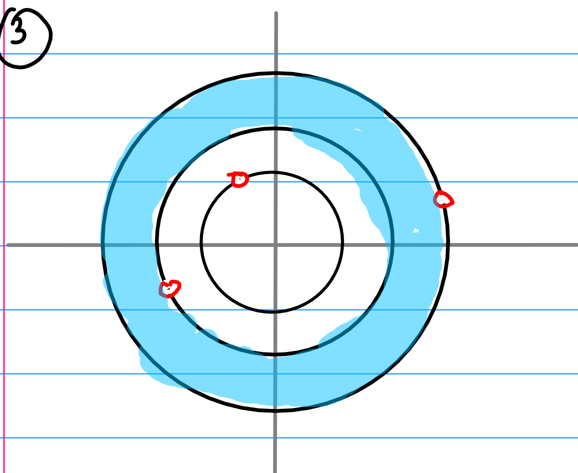
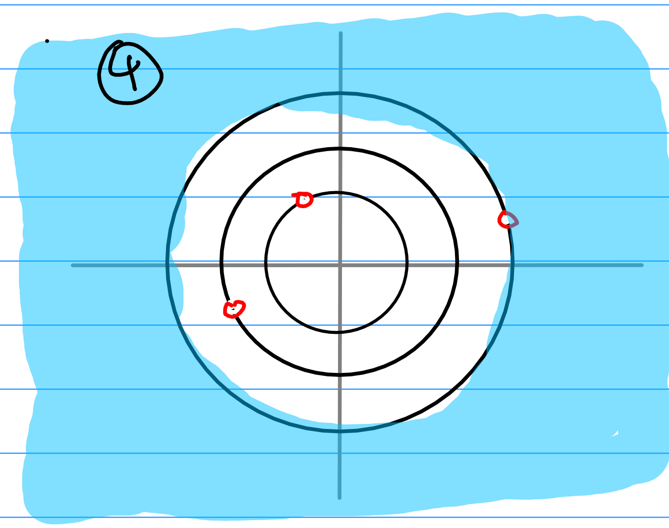
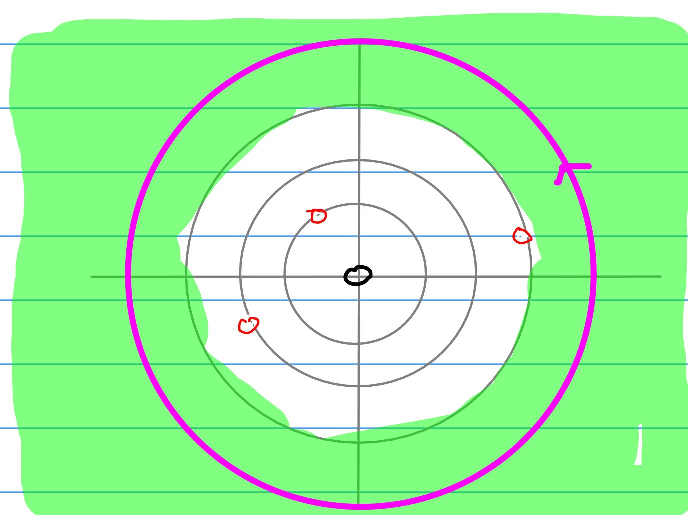


4

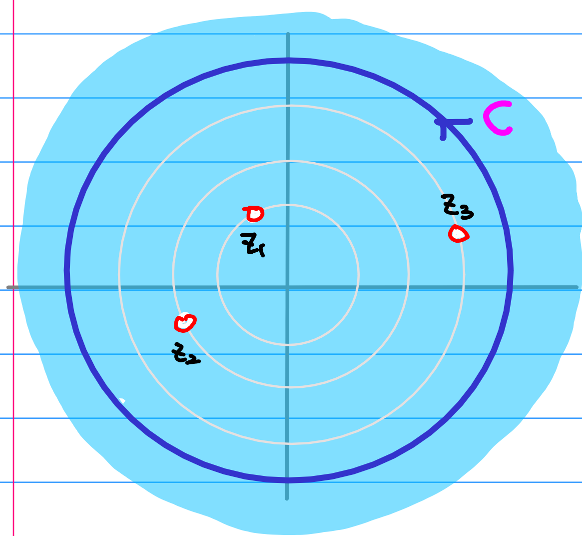


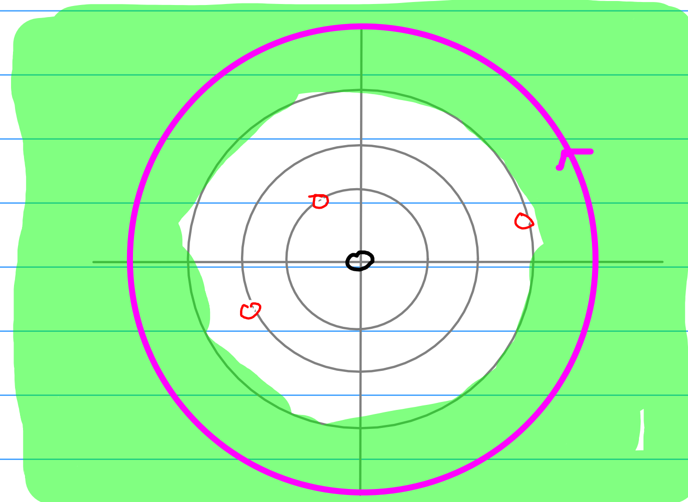


$$x[n] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$$

$$= \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right)$$

z-transform





$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} \, z^n}$$

$$\boxed{X(z) = \sum_{k=0}^{\infty} x_k \, z^{-k}}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\, dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$x_n = \frac{1}{2\pi i}\oint_C X(z)\, z^{n-1}\, dz = \sum_k \mathrm{Res}\left(X(z)\, z^{n-1}, z_k\right)$$

Poles z_k

$n \geq 0$ z_1, z_2, z_3, 0

$n < 0$ z_1, z_2, z_3

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

Complex Variables and Ap
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

D_1 : $|z| < 1$

D_2 : $1 < |z| < 2$

D_3 : $2 < |z|$

① D_1 $\quad |z| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(2^{-n-1} - 1\right) z^n \quad |z|<1$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

③ D_3 $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

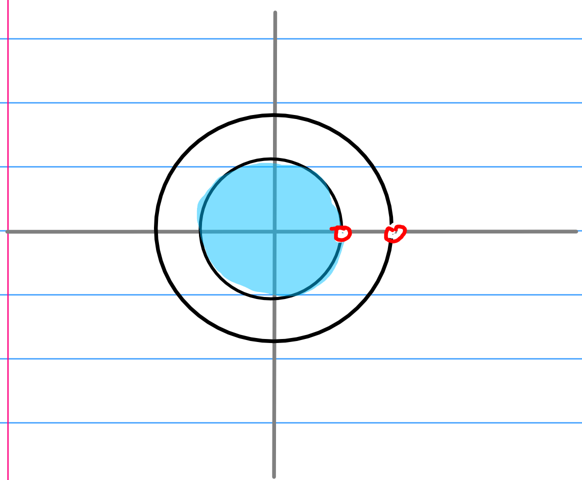
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

① D_1 $\quad |z|<1, \quad \left|\frac{z}{2}\right|<1$

$$\frac{f(z)}{z^{n+1}} = \frac{-1}{(z-1)(z-2)z^{n+1}}$$

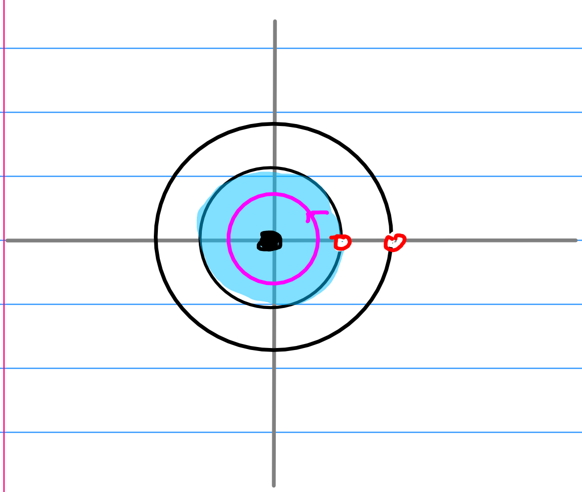


$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty}\left(2^{-n-1}-1\right)z^n \quad |z|<1$$

$$a_n = \frac{f(z)}{z^{n+1}} = \frac{1}{(z-1)(z-2)z^{n+1}} \qquad \frac{1}{z-1} - \frac{1}{z-2}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$n \geq 0$ then the pole $z=0$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{d}{dz}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)\left((z-1)^{-2} - (z-2)^{-2}\right)$$

$$\frac{d^2}{dz^2}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)\left((z-1)^{-3} - (z-2)^{-3}\right)$$

$$\frac{d^3}{dz^3}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)(-3)\left((z-1)^{-4} - (z-2)^{-4}\right)$$

$$\frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n\, n!\left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0}\left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n\left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

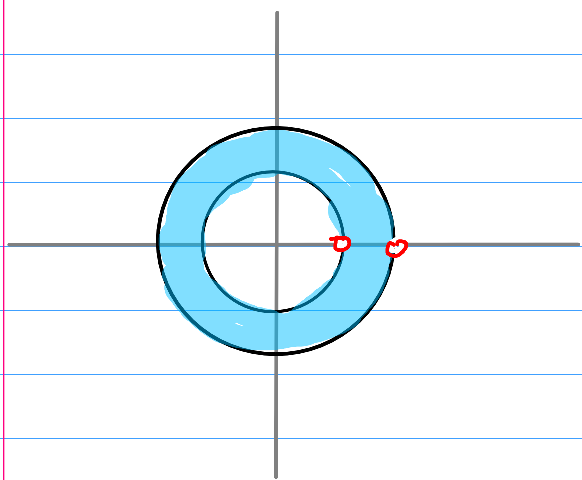
$$= -1 + 2^{-n-1}$$

$$a_n = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \left(2^{-n-1} - 1\right) z^n$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$



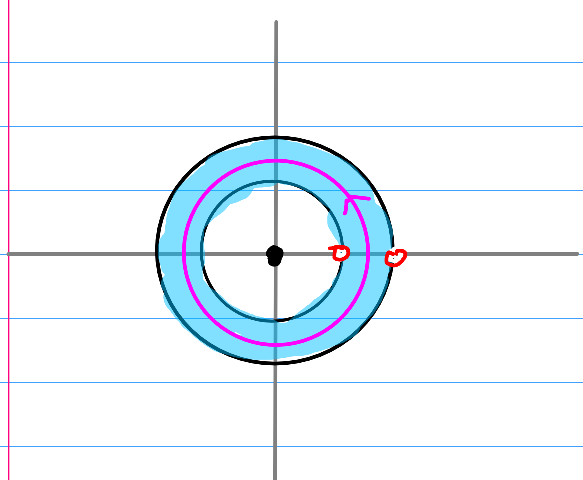
$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$a_n = \sum_{k=1}^{M} \operatorname{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_n)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n} \left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

$$= -1 + 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

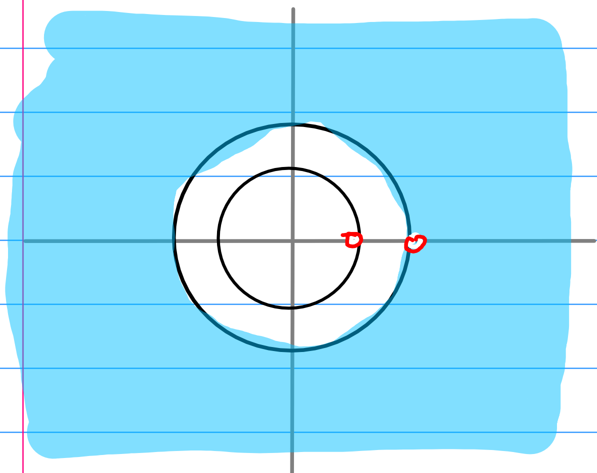
$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
1	1	1	2^{-1}	2^{-2}	2^{-3}	

$$\begin{cases} a_n = 2^{-n-1} & n \geq 0 \\ a_n = 1 & n < 0 \end{cases} \qquad \begin{cases} 2^{-n-1} \, z^n \\ z^{-n} \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

③ D_3 $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$



$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

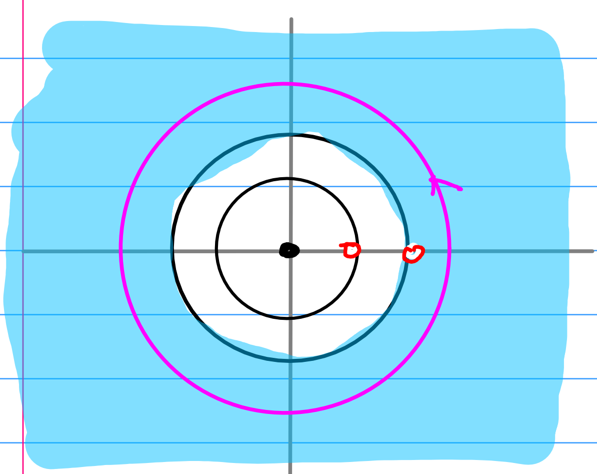
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_n)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right)$$



$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z\to 1} (z-1)\frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right) = \lim_{z\to 2} (z-2)\frac{-1}{(z-1)(z-2)z^{n+1}} = -\frac{1}{2^{n+1}}$$

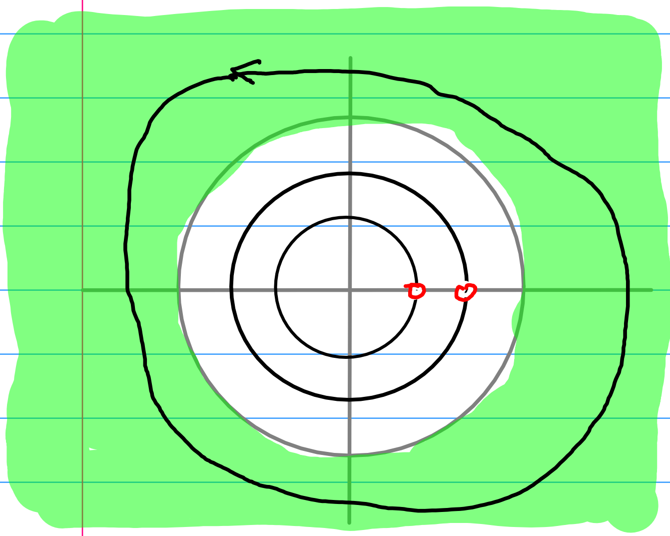
$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
-2^2	-2	-1	-2^{-1}	-2^{-2}	-2^{-3}	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 2\right)$
$1-2^2$	$1-2$	0	0	0	0	

$$a_n = 1 - 2^{-n+1} \quad n < 0$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \sum_{n=-1}^{-\infty} (1-2^{-n+1}) z^n = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



$x[n]$

$$= \frac{1}{2\pi i} \int_C \boxed{X(z)\, z^{n-1}}\, dz$$

$$= \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, z_j \right)$$

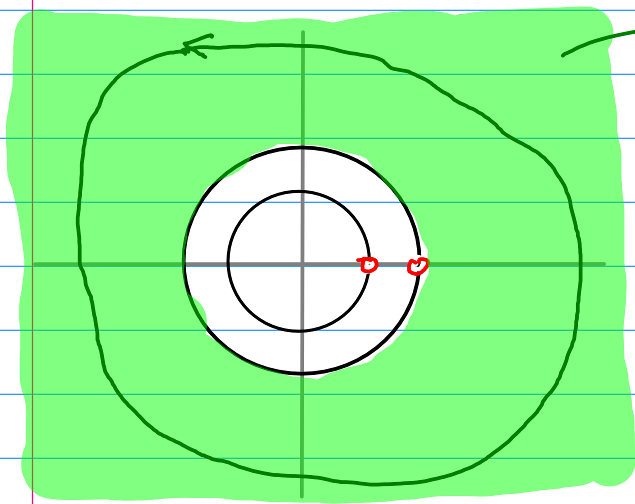
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z)\, z^{n-1} = \frac{-1}{(z-1)(z-2)}\, z^{n-1}$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 1 \right) = (z-1)\frac{-1}{(z-1)(z-2)}\, z^{n-1} \Big|_{z=1} = 1$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 2 \right) = (z-2)\frac{-1}{(z-1)(z-2)}\, z^{n-1} \Big|_{z=2} = -2^{n-1}$$

$$x[n] = 1 - 2^{n-1}$$



ROC (Region of Convergence)

$|z| > 2 \Rightarrow \frac{2}{|z|} < 1$

$\left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1 - \frac{2}{z}}$

$|z| > 2 \Rightarrow \frac{1}{|z|} < 1$

$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1 - \frac{1}{z}}$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\,\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\,\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots$$
$$+ \frac{1}{2}\left\{ \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots \right\} \xrightarrow[\text{converge}]{} \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

$$(1-2^0)z^{-1} + (1-2^1)z^{-2} + (1-2^2)z^{-3} + \cdots \xrightarrow[\text{converge}]{} \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

$$x[n] = 1 - 2^{n-1} \longleftrightarrow X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$