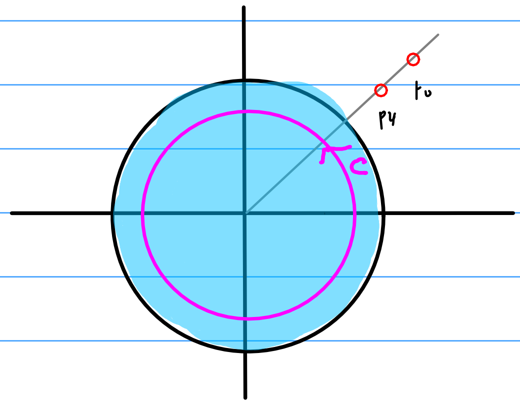
Laurent Series and z-Transform Examples

20170926

L.S at $z=0$



$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

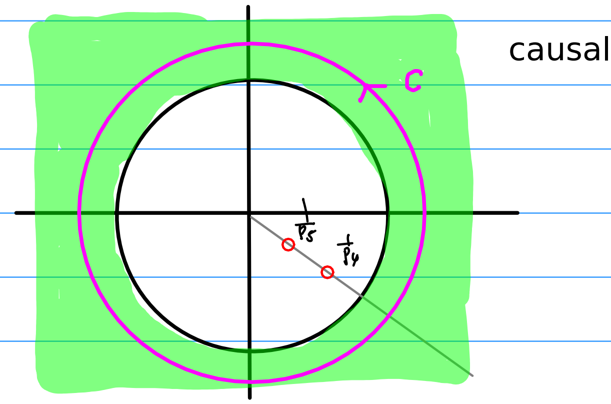
$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} \, dz$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

Poles z_k

$n \geq 0$	$z_1, z_2, z_3, 0$
$n < 0$	z_1, z_2, z_3

Z.T.



$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

$$x_n = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} \, dz$$

$$= \sum_k \text{Res}\left(X(z) z^{n-1}, z_k\right)$$

Poles z_k

$n > 0$	z_1, z_2, z_3
$n \leq 0$	$z_1, z_2, z_3, 0$

Z-transform

$\boxed{z_m = 0}$

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi i}\oint_C f(z)\, z^{n-1}\, dz \\ &= \sum_k \mathrm{Res}\left(f(z)\, z^{n-1},\, z_k\right) \end{aligned}}$$

$n > 0$ $\quad z_k : \{\text{poles of } f(z)\}$

$n = 0$ $\quad z_k : \{\text{poles of } f(z)\} + \{z = 0\}$

$$z^{n-1} = z^{-1} = \frac{1}{z}$$

$x[n]$ includes $u[n]$ $\rightarrow$ $X[z]$ contains z on its numerator

Also, think about modified partial fraction $\dfrac{X[z]}{z}$

Laurent Expansion

expansion at z_m

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_m)^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}},\, z_k\right) \end{aligned}}$$

$\boxed{z_m = 0}$

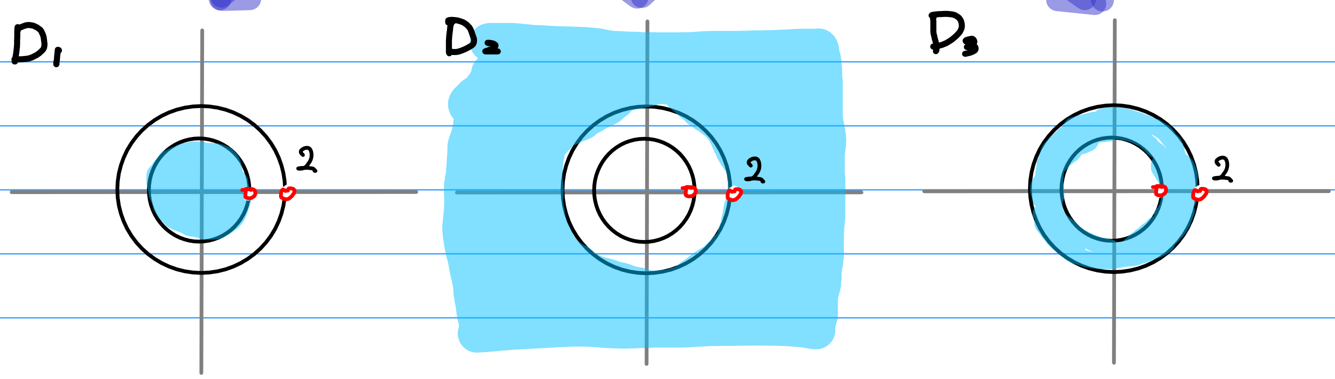
$$\boxed{\begin{aligned} a_n^{\{0\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}},\, z_k\right) \end{aligned}}$$

$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i}\oint_C f(z)\, z^{n-1}\, dz \\ &= \sum_k \mathrm{Res}\left(f(z)\, z^{n-1},\, z_k\right) \end{aligned}$$

$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{-n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{-n+1}},\, z_k\right) \end{aligned}$$

L.S. first

$$\boxed{f(z) = \frac{-1}{(z-1)(z-2)}}$$



a_n ?	a_n ?	a_n ?
$\parallel$	$\parallel$	$\parallel$
x_n	x_n	x_n
$X(z)$?	$X(z)$?	$X(z)$?

$$\boxed{X(z) = ?}$$

L.S. first

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$

Complex Variables and Ap
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

D_1 : $|z| < 1$

D_2 : $2 < |z|$

D_3 : $1 < |z| < 2$

I

D_1 $\quad |z| < 1$ $\qquad \left|\frac{z}{1}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(\left(\frac{1}{2}\right)^{n+1} - 1\right) z^n \qquad |z|<1$$

II

D_2 $\quad 2 < |z|$ $\qquad \left|\frac{1}{z}\right| < 1, \quad \left|\frac{2}{z}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} = \sum_{n=-1}^{-\infty} \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) z^n$$

III

D_3 $\quad 1 < |z| < 2$ $\quad \Rightarrow \quad \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{\frac{1}{z}}{1-\left(\frac{1}{z}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

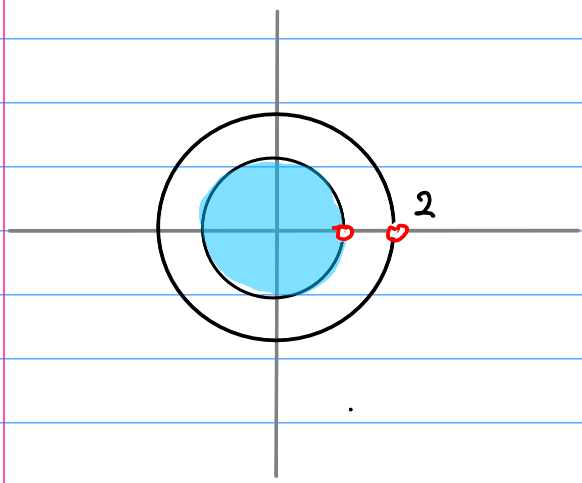
$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$(n < 1)$ $\qquad (n \geq 0)$

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$

L.S. first

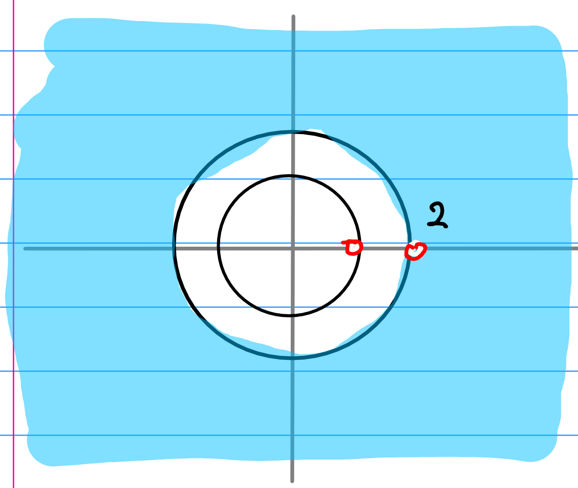
① D_1 $|z| < 1$ causal $a_n = 0 \quad (n \le 0)$



$$\boxed{f(z) = \frac{-1}{(z-1)(z-2)}}$$

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

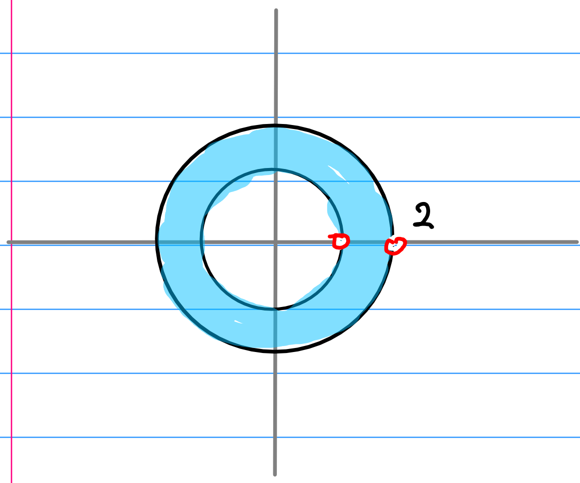
② D_3 $|z| > 2$ anti-causal $a_n = 0 \quad (n \geq 0)$



$$\boxed{f(z) = \frac{-1}{(z-1)(z-2)}}$$

$$a_n = \begin{cases} 0 & (n \geq 0) \\ 1 - \left(\frac{1}{2}\right)^{n+1} & (n < 0) \end{cases}$$

③ D_2 $1 < |z| < 2$ two-sided



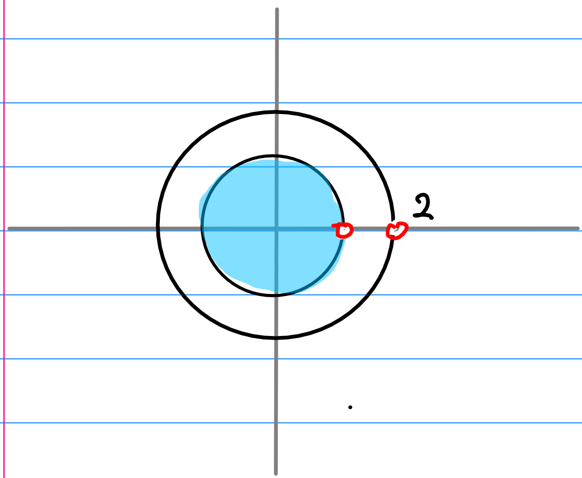
$$\boxed{f(z) = \frac{-1}{(z-1)(z-2)}}$$

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

L.S. first

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$

Ⓘ D_1 $|z| < 1$ causal $a_n = 0 \quad (n \le 0)$



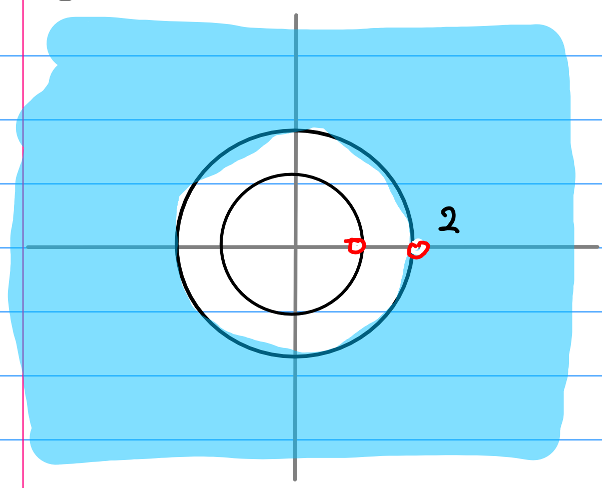
$$\left|\frac{z}{1}\right| < 1 \qquad \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} (1)^n z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{n+1} - 1\right] z^n$$

Ⓘ D_3 $|z| > 2$ anti-causal $a_n = 0 \quad (n \ge 0)$



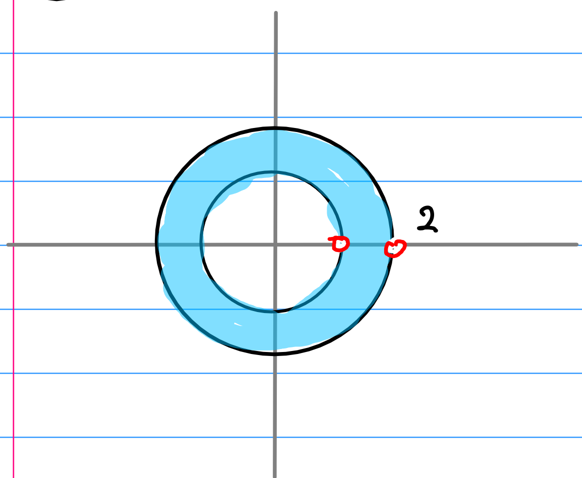
$$\left|\frac{1}{z}\right| < 1 \qquad \left|\frac{2}{z}\right| < 1$$

$$f(z) = \frac{\frac{1}{z}}{1-\left(\frac{1}{z}\right)} - \frac{\frac{2}{z}}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} 1^n z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1}$$

$$= \sum_{n=-1}^{-\infty} \left(1 - \left(\frac{1}{2}\right)^{n+1}\right) z^n$$

Ⓘ D_2 $1 < |z| < 2$ two-sided



$$\left|\frac{1}{z}\right| < 1 \qquad \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{\frac{1}{z}}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$

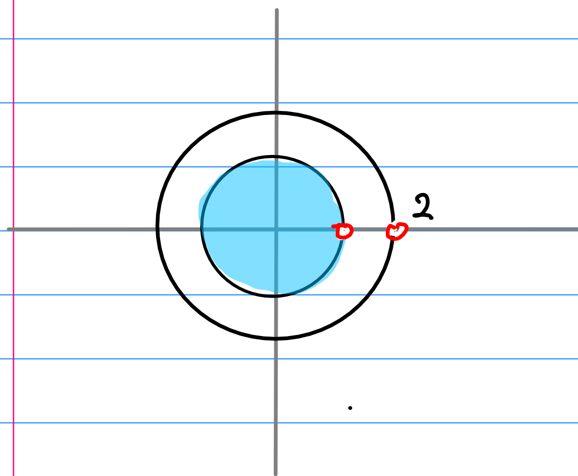
$$= \sum_{n=0}^{\infty} 1^n z^{-n-1} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n$$

$$= \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \cdot z^n$$

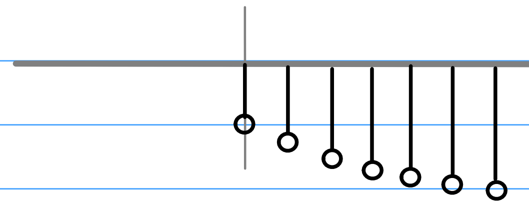
L.S. first

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$

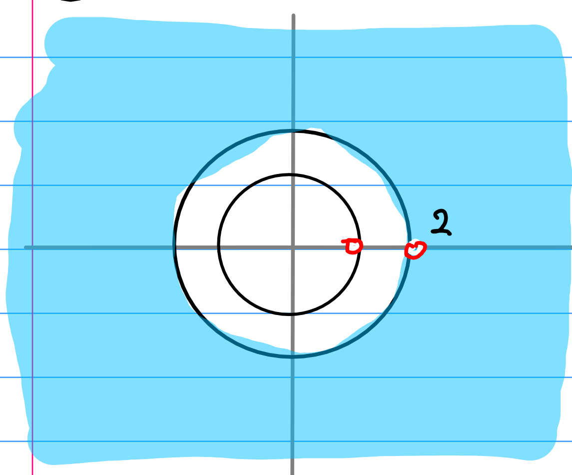
Ⅰ D_1 $|z| < 1$ causal $a_n = 0 \quad (n \leq 0)$



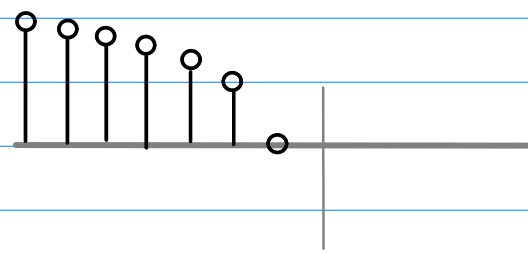
$$f(z) = \sum_{n=0}^{\infty} \left[\left(\tfrac{1}{2}\right)^{n+1} - 1 \right] z^n$$



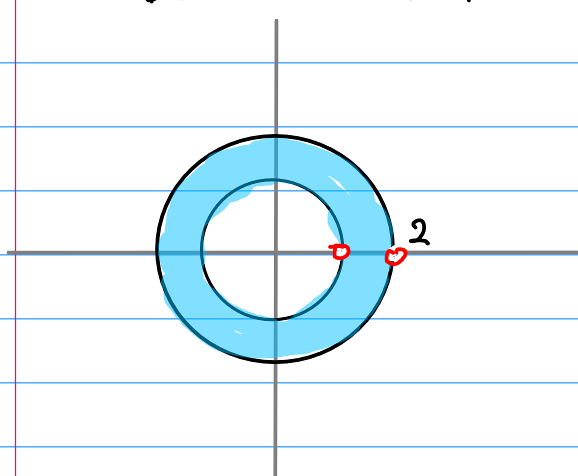
Ⅱ D_3 $|z| > 2$ anti-causal $a_n = 0 \quad (n \geq 0)$



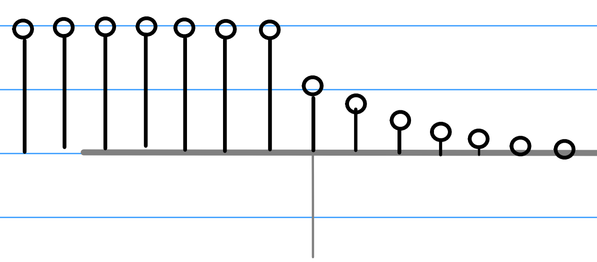
$$f(z) = \sum_{n=-1}^{-\infty} \left[1 - \left(\tfrac{1}{2}\right)^{n+1} \right] z^n$$



Ⅲ D_2 $1 < |z| < 2$ two-sided



$$f(z) = \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{n+1} \cdot z^n$$



$$f(z) = \frac{-1}{(z-1)(z-2)} \qquad X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

L.S. first

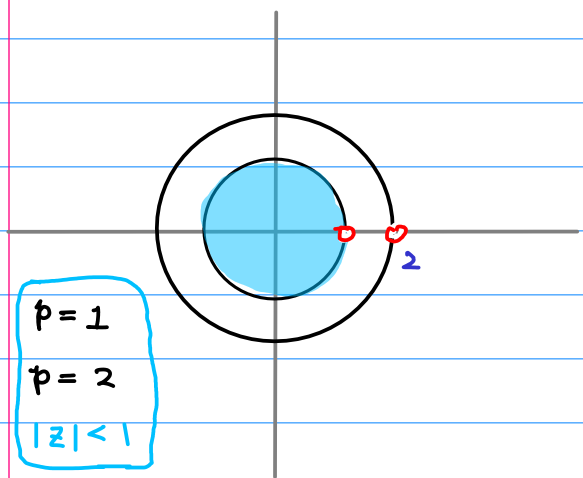
Ⓘ − 1

Ⓘ D_1 $\quad |z|<1 \quad \left[\left|\frac{z}{1}\right|<1, \quad \left|\frac{z}{2}\right|<1 \right]$

$p=1$

$p=2$

$|z|<1$

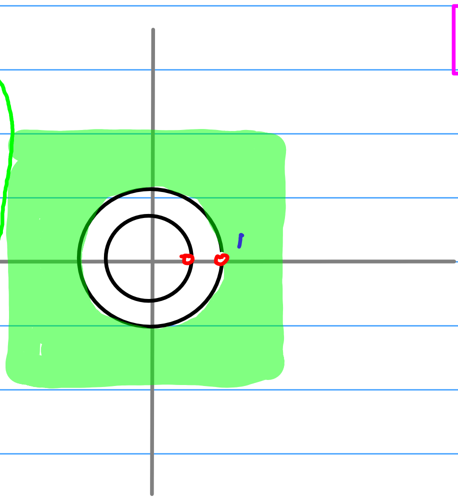


$$\begin{aligned}
f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
&= \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)} \\
&= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\
&= \sum_{n=0}^{\infty} \left[2^{-n-1} - 1 \right] z^n \qquad |z|<1 \\
&= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n \qquad |z|<1
\end{aligned}$$

$p=1$

$p=0.5$

$|z|>1$



$$\begin{aligned}
\boxed{X(z) = f(z^{-1})} &= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{n+1} - 1 \right] z^{-n} \qquad |z|>2 \\
&= \sum_{n=0}^{\infty} 2^{-n-1} z^{-n} - \sum_{n=0}^{\infty} 1 \cdot z^{-n} \\
&= \frac{\frac{1}{2}}{1-\left(\frac{1}{2z}\right)} - \frac{1}{1-\left(\frac{1}{z}\right)} \\
&= \frac{0.5z}{z-0.5} - \frac{z}{z-1} \\
&= \frac{0.5z - 0.5 - z + 0.5}{(z-0.5)(z-1)} \cdot z \\
&= \frac{-0.5z^2}{(z-1)(z-0.5)}
\end{aligned}$$

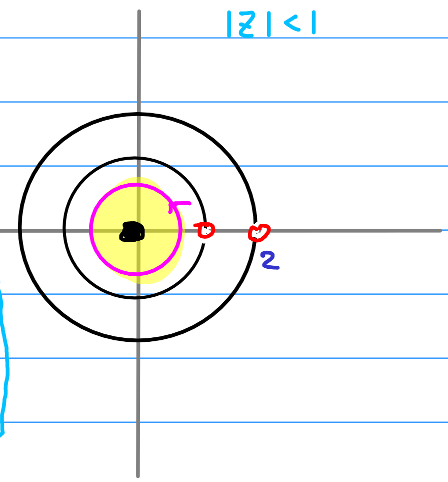
$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$f(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = X(z)$$

L.S. first

Ⅰ − 2

$$a_n \Leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$



$p=1$

$p=2$

$|z|<1$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$= \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$\frac{f(z)}{z^{n+1}} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)\frac{1}{z^{n+1}}$$

$n \geq 0$ then the pole $z=0$

$$\boxed{\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z) \quad (\text{order } n)}$$

$$\frac{d}{dz}\left((z-1)^{-1}-(z-2)^{-1}\right) = (-1)\left((z-1)^{-2}-(z-2)^{-2}\right)$$

$$\frac{d^2}{dz^2}\left((z-1)^{-1}-(z-2)^{-1}\right) = (-1)(-2)\left((z-1)^{-3}-(z-2)^{-3}\right)$$

$$\frac{d^3}{dz^3}\left((z-1)^{-1}-(z-2)^{-1}\right) = (-1)(-2)(-3)\left((z-1)^{-4}-(z-2)^{-4}\right)$$

$$\frac{d^n}{dz^n}\left((z-1)^{-1}-(z-2)^{-1}\right) = (-1)^n\, n!\left((z-1)^{-n-1}-(z-2)^{-n-1}\right)$$

$$\frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n} z^{n+1}\left((z-1)^{-1}-(z-2)^{-1}\right)\frac{1}{z^{n+1}} = \frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}\left((z-1)^{-1}-(z-2)^{-1}\right)$$

$$= \frac{1}{n!}\lim_{z\to 0}(-1)^n\, n!\left((z-1)^{-n-1}-(z-2)^{-n-1}\right)$$

$$= (-1)^n \lim_{z\to 0}\left((z-1)^{-n-1}-(z-2)^{-n-1}\right)$$

$$= (-1)^n\left((-1)^{-n-1}-(-2)^{-n-1}\right)$$

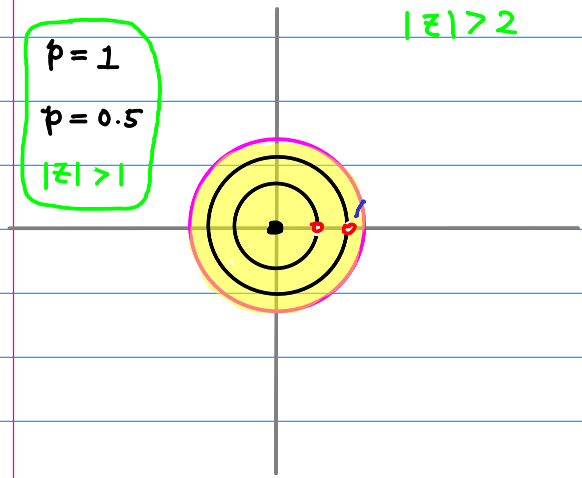
$$= -1 + 2^{-n-1}$$

$$a_n = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$f(z) = \sum_{n=n_0}^{\infty} a_n z^n = \sum_{n=0}^{\infty}\left(\left(\frac{1}{2}\right)^{n+1}-1\right)z^n$$

L.S. first Ⅰ − 3

$$x_n \Leftarrow X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$



$$x[n] = \frac{1}{2\pi i}\int_C \boxed{X(z)\, z^{n-1}}\, dz$$

$$= \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, z_j\right)$$

$$X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$X(z)\, z^{n-1} = \frac{-0.5z^2}{(z-1)(z-0.5)}\, z^{n-1}$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 1\right) = (z-1)\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}\Bigg|_{z=1} = -1$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, \frac{1}{2}\right) = (z-0.5)\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}\Bigg|_{z=0.5} = +\left(\frac{1}{2}\right)^{n+1}$$

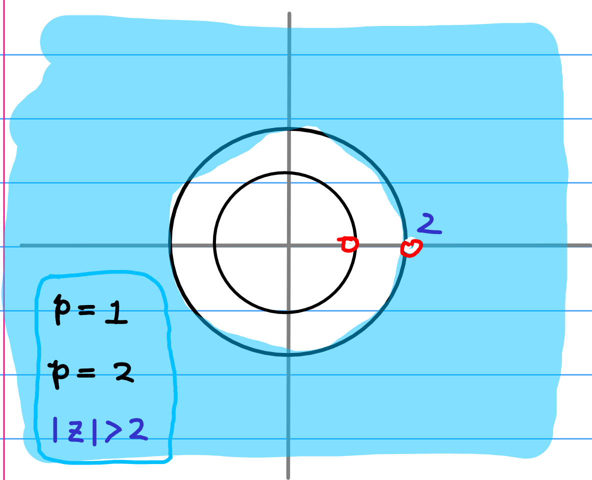
$$x[n] = \left(\frac{1}{2}\right)^{n+1} - 1$$

$$f(z) = \frac{-1}{(z-1)(z-2)} \qquad X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

L.S. first

II -1

II D_2 $\quad |z| > 2 \quad \left[\left|\frac{1}{z}\right| < 1 , \quad \left|\frac{2}{z}\right| < 1 \right]$



$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

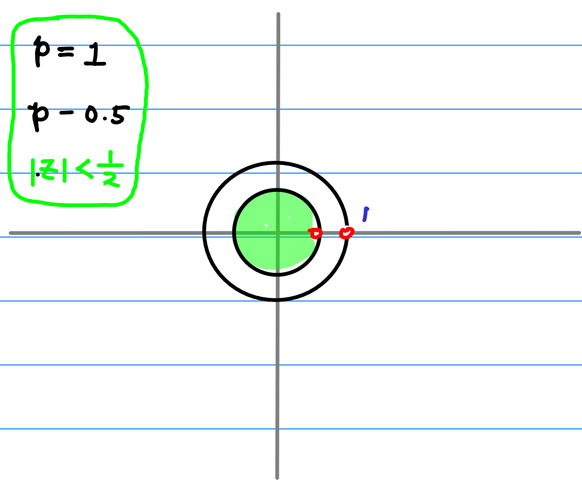
$$= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)\left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)\left(\frac{2}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1}$$

$$= \sum_{n=1}^{\infty} z^{-n} - \sum_{n=1}^{\infty} 2^{n-1} z^{-n}$$

$$= \sum_{n=-1}^{\infty} \left[1 - 2^{-n-1}\right] z^n = \sum_{n=-1}^{\infty} \left[1 - \left(\frac{1}{2}\right)^{n+1}\right] z^n$$



$$\boxed{X(z) = f(z^{-1})} = \sum_{n=-1}^{\infty} \left[1 - 2^{-n-1}\right] z^{-n}$$

$$= \sum_{n=1}^{\infty} \left[1 - 2^{n-1}\right] z^n$$

$$= \sum_{n=1}^{\infty} 1 \cdot z^n - \sum_{n=1}^{\infty} 2^{n-1} z^n$$

$$= \frac{z}{1-z} - \frac{z}{1-2z}$$

$$= -\frac{z}{z-1} + \frac{0.5z}{z-0.5}$$

$$= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} z$$

$$= \frac{-0.5z^2}{(z-1)(z-0.5)}$$

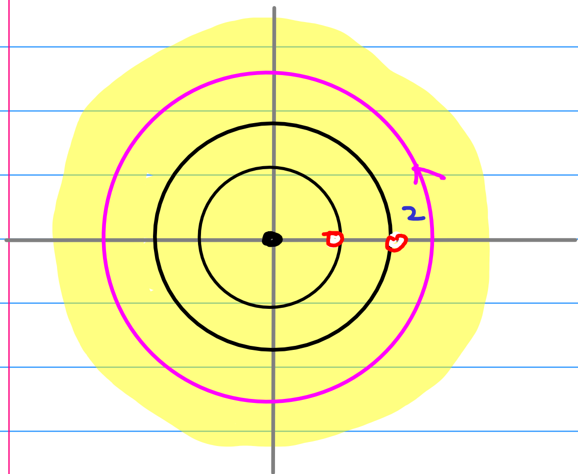
$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$f(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = X(z)$$

L.S. first

Ⅱ -2

$$a_n \Leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$= \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1)\frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right) = \lim_{z \to 2} (z-2)\frac{-1}{(z-1)(z-2)z^{n+1}} = -\frac{1}{2^{n+1}}$$

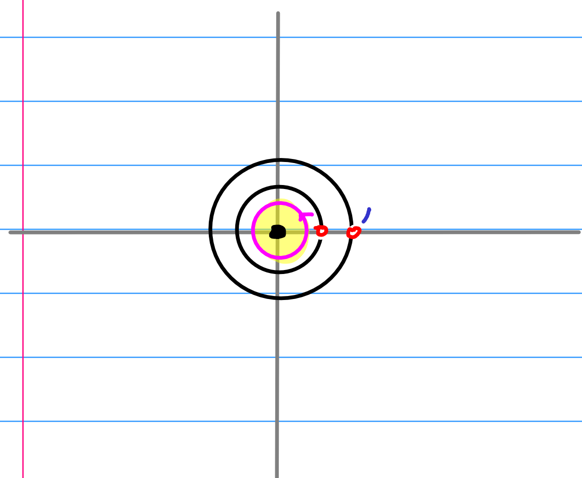
$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
-2^2	-2^1	-2^0	-2^{-1}	-2^{-2}	-2^{-3}	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 2\right)$
$1-2^2$	$1-2$	0	0	0	0	
$-(-3)-1$	$-(-2)-1$	$-(-1)-1$				

$$a_n = 1 - 2^{-n-1} \quad n < 0$$

$$f(z) = \sum_{n=-1}^{-\infty} (1 - 2^{-n-1}) z^n = \sum_{n=-1}^{-\infty} \left(1 - \left(\tfrac{1}{2}\right)^{n+1}\right) z^n$$

$$= \sum_{n=1}^{\infty} (1 - 2^{n-1}) z^{-n}$$

$$x_n \leftarrow X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$



$$x[n] = \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)z^{n-1}}, z_j\right)$$

$$= \mathrm{Res}\left(\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, 0\right)$$

$$= \mathrm{Res}\left(z^{n+1}\left(\frac{1}{z-0.5}-\frac{1}{z-1}\right), 0\right)$$

$$\boxed{\frac{1}{n!}\lim_{z\to z_0}\frac{d^n}{dz^n}(z-z_0)^{n+1}f(z) \quad \text{(n+1)th order}}\ \text{pole at } z=0$$

$n=-1$: $\qquad (-2^0+1)$

$n=-2$: $\quad z^1 z^{-1}\left((z-0.5)^{-1}-(z-1)^{-1}\right) = \left((z-0.5)^{-1}-(z-1)^{-1}\right) \qquad 0!(-2^1+1)$

$n=-3$: $\quad \frac{d}{dz} z^2 z^{-2}\left((z-0.5)^{-1}-(z-1)^{-1}\right) = (-1)\left((z-0.5)^{-2}-(z-1)^{-2}\right) \qquad 1!(-2^2+1)$

$n=-4$: $\quad \frac{d^2}{dz^2} z^3 z^{-3}\left((z-0.5)^{-1}-(z-1)^{-1}\right) = (-1)(-2)\left((z-0.5)^{-3}-(z-1)^{-3}\right) \qquad 2!(-2^3+1)$

$n=-k$: $\quad \frac{d^{k-2}}{dz^{k-2}} z^{k-1} z^{-k+1}\left((z-0.5)^{-1}-(z-1)^{-1}\right) = (-1)(-2)\cdots(-(k-2))\left((z-0.5)^{-k+1}-(z-1)^{-k+1}\right) \qquad (k-2)!(-2^{k-1}+1)$

$$x_{-k} = \frac{1}{(k-2)!}\lim_{z\to 0}\frac{d^{k-2}}{dz^{k-2}} z^{k-1}z^{-k+1}\left((z-0.5)^{-1}-(z-1)^{-1}\right) = (-2^{k-1}+1) \qquad (k\geq 1) \quad (n\leq -1)$$

$$x_n = (-2^{-n-1}+1) \qquad (n<0)$$

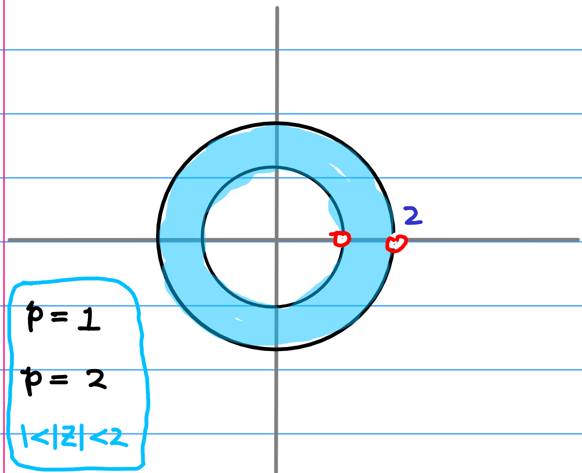
$$f(z) = \sum_{n=-1}^{-\infty}(1-2^{-n-1})z^n = \sum_{n=-1}^{-\infty}\left(1-\left(\tfrac{1}{2}\right)^{n+1}\right)z^n$$

$$= \sum_{n=1}^{\infty}(1-2^{n-1})z^{-n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)} \qquad X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

L.S. first Ⅲ -1

Ⅲ D_3 $\quad 1 < |z| < 2 \quad \left[\, \left|\frac{1}{z}\right| < 1 ,\quad \left|\frac{z}{2}\right| < 1 \,\right]$



$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$

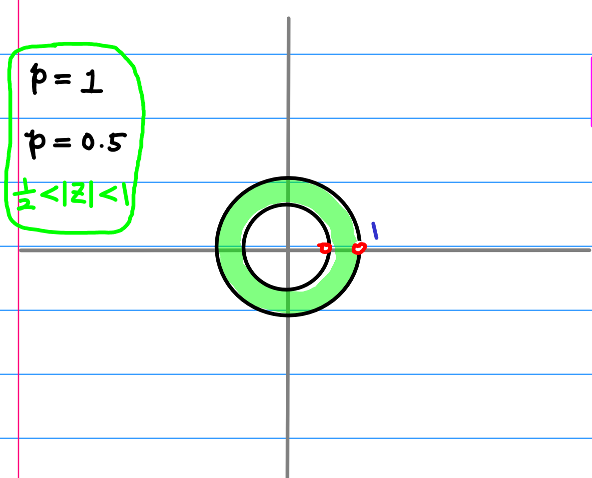
$$= \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$$= \sum_{n=-\infty}^{-1} z^n + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$[\; 1 \ (n<0) ,\quad 2^{-n-1} \ (n \geq 0) \;]$



$$\boxed{X(z) = f(z^{-1})} = \sum_{n=-\infty}^{-1} z^{-n} + \sum_{n=0}^{\infty}\frac{z^{-n}}{2^{n+1}}$$

$$= \boxed{\sum_{n=1}^{\infty} z^n} + \sum_{n=0}^{\infty}\frac{z^{-n}}{2^{n+1}}$$

$$= \frac{z}{1-z} - \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)}$$

$$= \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$

$$= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} \cdot z$$

$$= \frac{-0.5z^2}{(z-1)(z-0.5)}$$

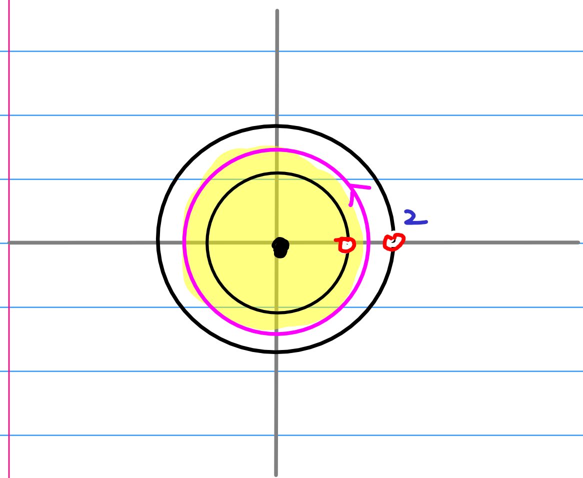
$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$f(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = X(z)$$

L.S. first

Ⅲ -2

$$a_n \leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

$$= \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$\boxed{\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)}$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n} \left((z-1)^{-1} - (z-2)^{-1} \right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1} \right)$$

$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1} \right)$$

$$= -1 + 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
1	1	1	2^{-1}	2^{-2}	2^{-3}	

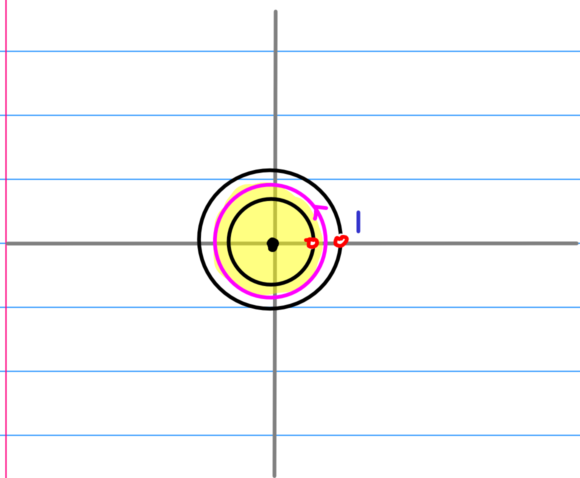
$$\begin{cases} a_n = 2^{-n-1} & n \geq 0 \\ a_n = 1 & n < 0 \end{cases} \qquad \begin{cases} 2^{-n-1} z^n \\ z^{-n} \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$x_n \leftarrow X(z) = \dfrac{-0.5z^2}{(z-1)(z-0.5)}$

L.S. first

Ⅲ -3



$$x[n] = \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)z^{n-1}}, z_j\right)$$

$$= \mathrm{Res}\left(\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, 0\right) + \mathrm{Res}\left(\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2}\right)$$

$$\mathrm{Res}\left(\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2}\right) = \lim_{z\to 0.5}\frac{-0.5z^{n+1}}{(z-1)} = \left(\frac{1}{2}\right)^{n+1} = 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, 0\right) = (-2^{-n-1}+1) \quad (n<0)$$

$(n<0)$ $\quad x_n = (-2^{-n-1}+1) - 2^{-n-1} = 1$

$(n\geq 0)$ $\quad x_n = 2^{-n-1}$

$(n<0)$ $\quad a_n = 1$

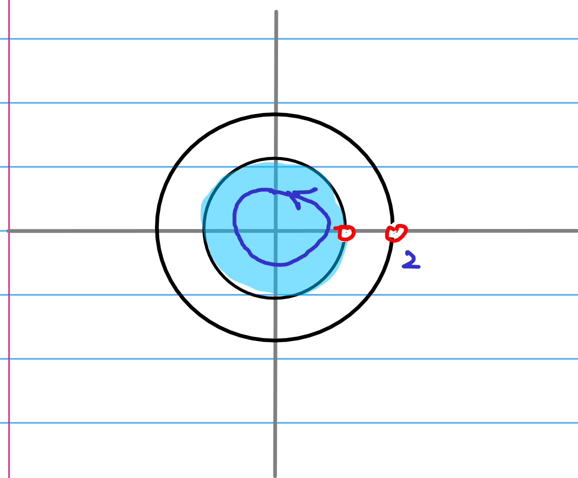
$(n\geq 0)$ $\quad a_n = 2^{-n-1}$

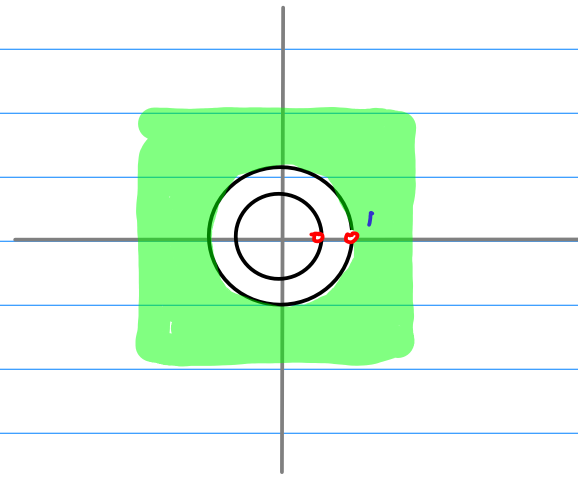
L.S. first

L.S. first

a_n, x_n using Res $\leftarrow$ $f(z) = \dfrac{-1}{(z-1)(z-2)}$

Ⅰ D_1 $\quad |z| < 1 \quad \left[\, \left|\frac{z}{1}\right| < 1 ,\quad \left|\frac{z}{2}\right| < 1 \,\right]$





$p = 0$	$\longrightarrow$	∞
$p = 1$	$\longrightarrow$	$p = 1$
$p = 2$	$\longrightarrow$	$p = 0.5$
∞	$\longrightarrow$	$p = 0$

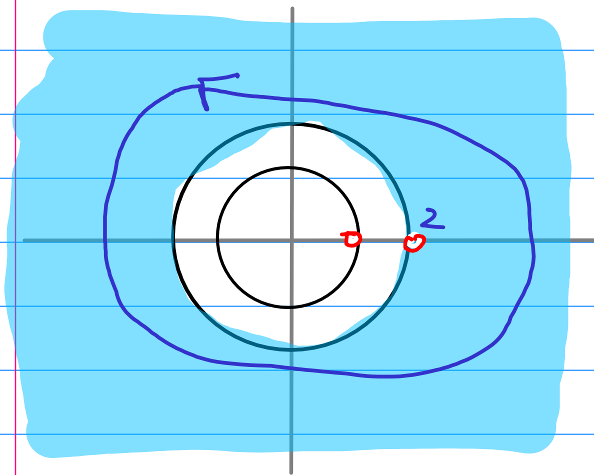
$$a_n = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}} , 0 \right)$$

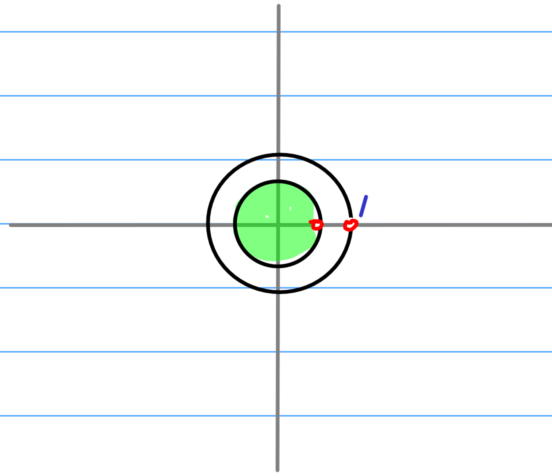
$$\begin{aligned} x_n &= \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)} , 0 \right) \\ &= \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)} , 1 \right) \\ &+ \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)} , \frac{1}{2} \right) \end{aligned}$$

L.S. first

a_n, x_n using **Res** ← $f(z) = \dfrac{-1}{(z-1)(z-2)}$

② D_2	$|z| > 2$	$\left[\left|\frac{1}{z}\right| < 1, \quad \left|\frac{2}{z}\right| < 1 \right]$





$p=0$	→	∞
$p=1$	→	$p=1$
$p=2$	→	$p=0.5$
∞	→	$p=0$

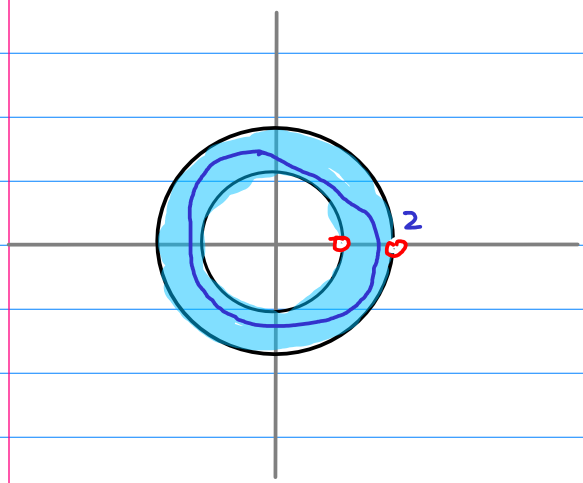
$$a_n = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) + \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) + \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right)$$

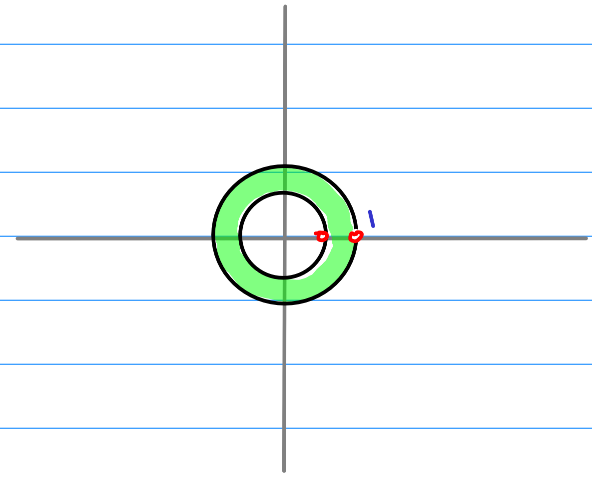
$$x_n = \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 0\right)$$

L.S. first

a_n, x_n using **Res** $\Leftarrow$ $f(z) = \dfrac{-1}{(z-1)(z-2)}$

$\boxed{\text{III} \quad D_3 \quad 1 < |z| < 2 \qquad \left[\left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1 \right]}$





$p = 0$	$\longrightarrow$	∞
$p = 1$	$\longrightarrow$	$p = 1$
$p = 2$	$\longrightarrow$	$p = 0.5$
∞	$\longrightarrow$	$p = 0$

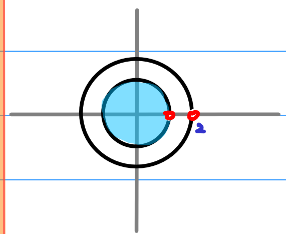
$$a_n = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) + \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$x_n = \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 0\right) + \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2}\right)$$

L.S. first

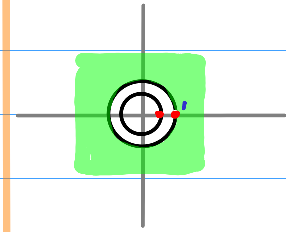
a_n, x_n using **Res** $\leftarrow$ $f(z) = \dfrac{-1}{(z-1)(z-2)}$

$(n \geq 0)$



$a_n = -1 + 2^{-n-1}$

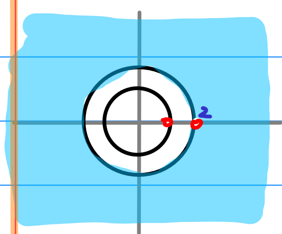
$$a_n = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$



$x_n = -1 + 2^{-n-1}$

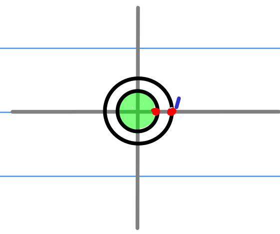
$$x_n = \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 1\right) + \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2}\right)$$

$(n < 0)$



$1 - 2^{-n-1}$

$$a_n = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) + \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) + \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right)$$

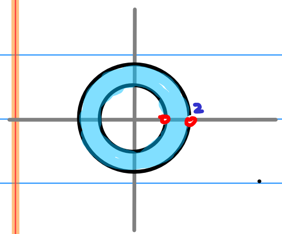


$1 - 2^{-n-1}$

$$x_n = \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 0\right)$$

$(n \geq 0)$

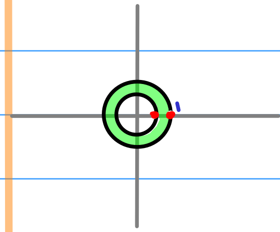
$(n < 0)$



$a_n = 2^{-n-1}$

$a_n = 1$

$$a_n = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) + \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$



$x_n = 2^{-n-1}$

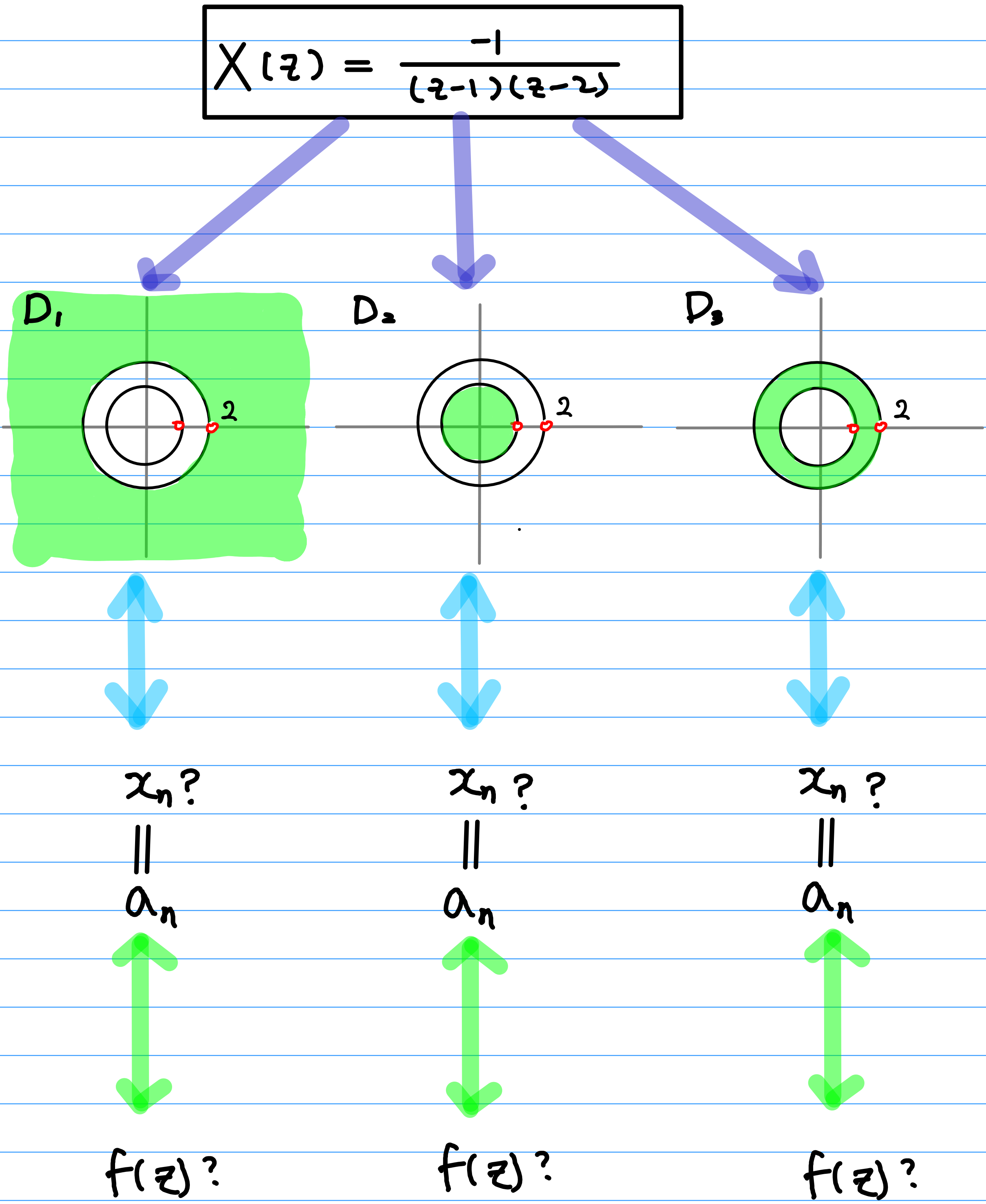
$x_n = 1$

$$x_n = \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, 0\right) + \mathrm{Res}\left(\frac{-0.5\, z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2}\right)$$

L.S. first

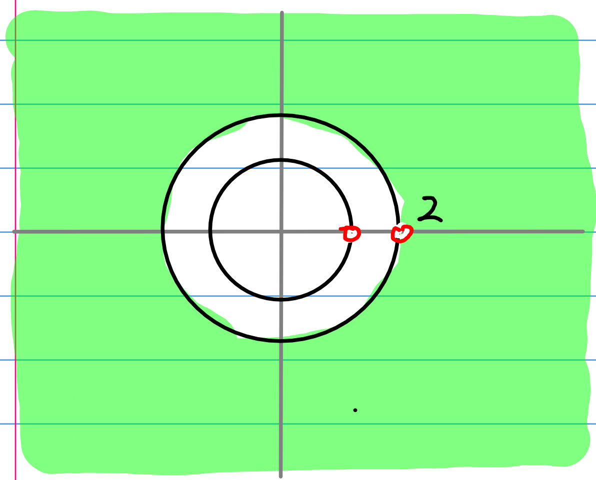
Z.T. first

$$X(z) = \frac{-1}{(z-1)(z-2)}$$



$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$ Z.T. first

① D_1 $\quad |z| > 2 \quad \left[\left|\frac{1}{z}\right| < 1, \quad \left|\frac{2}{z}\right| < 1 \right]$



$|z| > \frac{1}{2} \Rightarrow \frac{1}{|2z|} < 1$

$$\left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{2}{z}}$$

$|z| > 1 \Rightarrow \frac{1}{|z|} < 1$

$$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{1}{z}}$$

ROC (Region of Convergence)

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{z}{z-1} - \frac{1}{z}\frac{z}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

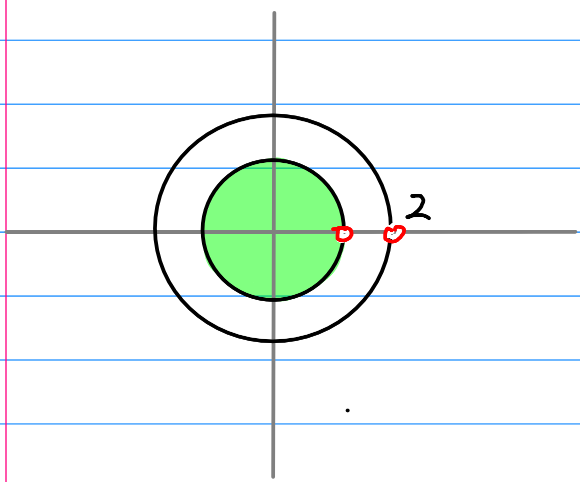
$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} = \sum_{n=1}^{\infty} (1-2^{n-1}) z^{-n}$$

$$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots$$
$$+ \frac{1}{2}\left\{ \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots \right\} \xrightarrow[\text{converge}]{} \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

$$(1-2^0)z^{-1} + (1-2^1)z^{-2} + (1-2^2)z^{-3} + \cdots \xrightarrow[\text{converge}]{} \frac{-1}{(z-1)(z-2)} \quad (|z|>2)$$

$$x[n] = 1 - 2^{n-1} \quad \longleftrightarrow \quad X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z|>2)$$

$\boxed{\text{II} \quad D_2 \quad |z|<1 \qquad \left[\left|\frac{z}{1}\right|<1, \quad \left|\frac{z}{2}\right|<1\right]}$



$|z|<2 \Rightarrow \frac{|z|}{2}<1$

$\left(\frac{z}{2}\right)^0 + \left(\frac{z}{2}\right)^1 + \left(\frac{z}{2}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{z}{2}}$

$|z|<1 \Rightarrow \frac{|z|}{1}<1$

$\left(\frac{z}{1}\right)^0 + \left(\frac{z}{1}\right)^1 + \left(\frac{z}{1}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{z}{1}}$

ROC (Region of onvergence)

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = -\frac{1}{1}\frac{1}{1-z} + \frac{1}{2}\frac{2}{2-z} = -\frac{1}{1}\frac{1}{1-\left(\frac{z}{1}\right)} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = \sum_{n=0}^{\infty}(-1+2^{-n-1})z^n$$

$$= \sum_{k=0}^{-\infty}(-1+2^{k-1})z^{-k} = \sum_{n=0}^{-\infty}(-1+2^{n-1})z^{-n}$$

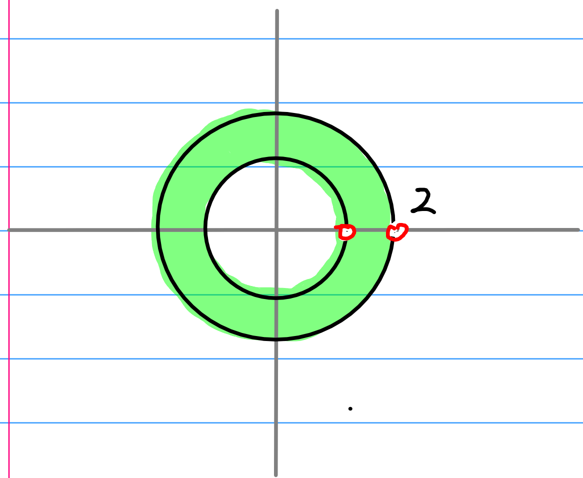
$$\begin{aligned} &-\left\{\left(\frac{z}{1}\right)+\left(\frac{z}{1}\right)^2+\left(\frac{z}{1}\right)^3+\cdots\right\} \\ &+\frac{1}{2}\left\{\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right\}\end{aligned} \xrightarrow[\text{converge}]{} \frac{1}{z-1}-\frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

$$(-1+2^{-2})z^1 + (-1+2^{-3})z^2 + (-1+2^{-4})z^3 + \cdots \xrightarrow[\text{converge}]{} \frac{-1}{(z-1)(z-2)} \quad (|z|<2)$$

$$x[n] = -1+2^{n-1} \longleftrightarrow X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z|>2)$$

$n \leq 0$

Ⅲ D_3 $\quad 1<|z|<2 \quad \left[\ \left|\frac{1}{z}\right|<1, \quad \left|\frac{z}{2}\right|<1\ \right]$



$|z|<2 \Rightarrow \frac{|z|}{2}<1$

$\left(\frac{z}{2}\right)^0+\left(\frac{z}{2}\right)^1+\left(\frac{z}{2}\right)^2+\cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{z}{2}}$

$|z|>1 \Rightarrow \frac{1}{|z|}<1$

$\left(\frac{1}{z}\right)^0+\left(\frac{1}{z}\right)^1+\left(\frac{1}{z}\right)^2+\cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{1}{z}}$

ROC (Region of Convergence)

$$f(z)=\frac{1}{z-1}-\frac{1}{z-2}=\frac{1}{z}\,\frac{z}{z-1}+\frac{1}{2}\,\frac{2}{z-2}=\frac{1}{z}\,\frac{1}{1-\left(\frac{1}{z}\right)}+\frac{1}{2}\,\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{k=0}^{-\infty}\frac{z^{-k}}{2^{-k+1}}=\sum_{n=1}^{\infty}z^{-n}+\sum_{n=0}^{-\infty}2^{n-1}z^{-n}$$

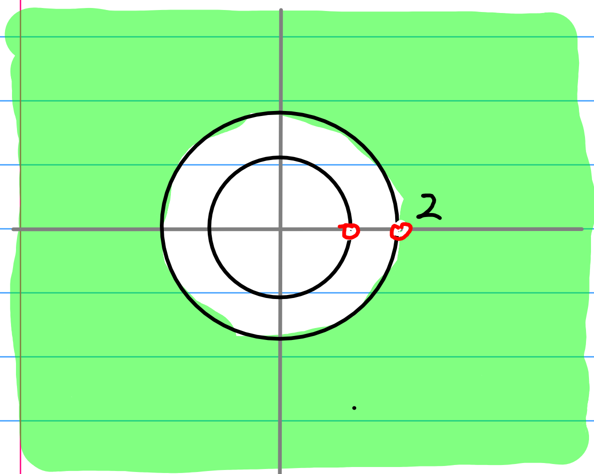
$$x_n=\begin{cases}1 & (n>0)\\ 2^{n-1} & (n\le 0)\end{cases}$$

$$\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots$$
$$+\frac{1}{2}\left\{\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right\} \xrightarrow[\text{converge}]{} \frac{1}{z-1}-\frac{1}{z-2}=\frac{-1}{(z-1)(z-2)}$$

Z.T. first

$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

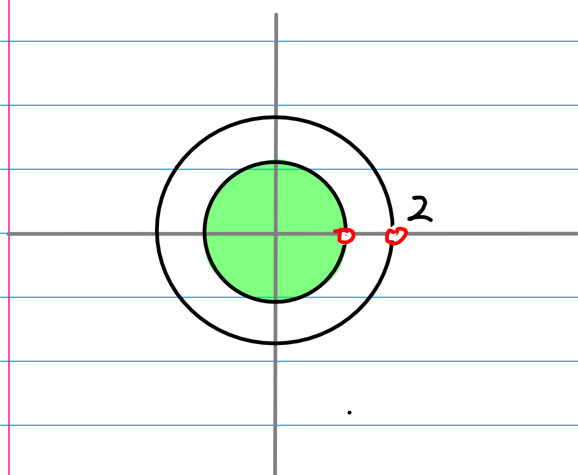
① D_1 $|z| > 2$ causal $x_n = 0 \quad (n \leq 0)$



$$\boxed{X(z) = \frac{-1}{(z-1)(z-2)}}$$

$$x_n = \begin{cases} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

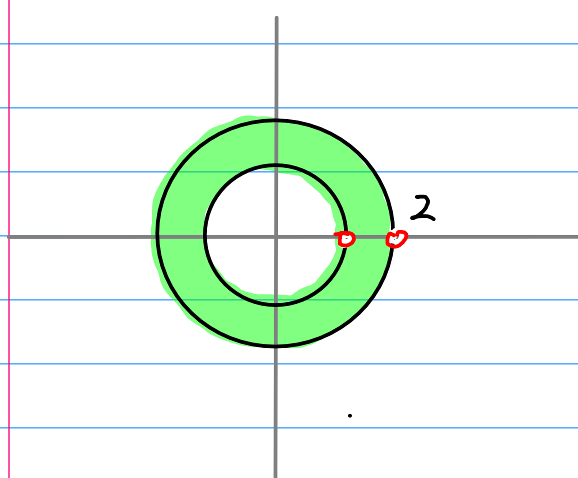
② D_3 $|z| < 1$ anti-causal $x_n = 0 \quad (n \geq 0)$



$$\boxed{X(z) = \frac{-1}{(z-1)(z-2)}}$$

$$x_n = \begin{cases} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{cases}$$

③ D_2 $1 < |z| < 2$ two-sided



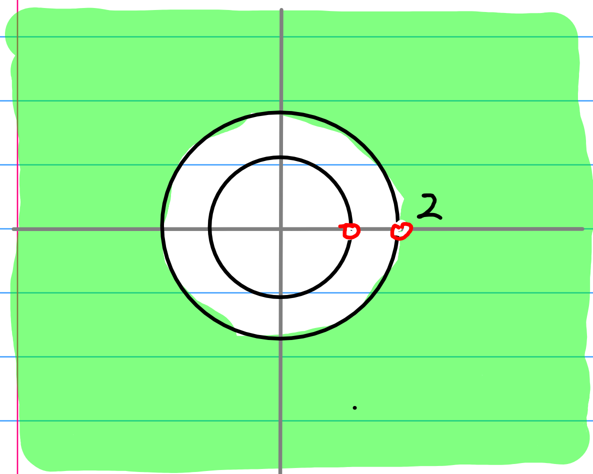
$$\boxed{X(z) = \frac{-1}{(z-1)(z-2)}}$$

$$x_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

Z.T. first

$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

① D_1 $|z| > 2$ causal $x_n = 0 \quad (n<0)$



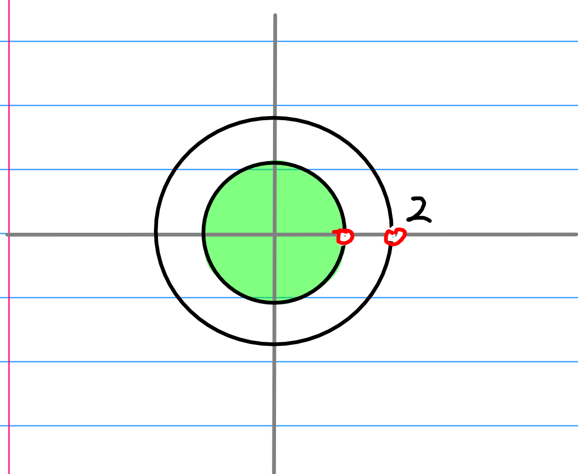
$$\left|\frac{1}{z}\right| < 1 \qquad \left|\frac{2}{z}\right| < 1$$

$$X(z) = \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n-1} - \sum_{n=0}^{\infty} 2^{n} z^{-n-1}$$

$$= \sum_{n=1}^{\infty} \left[1 - 2^{-n+1}\right] z^{-n}$$

② D_3 $|z| < 1$ anti-causal $x_n = 0 \quad (n>0)$



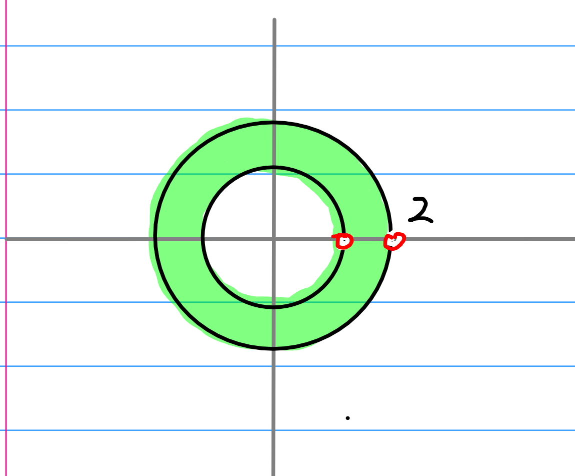
$$\left|\frac{z}{1}\right| < 1 \qquad \left|\frac{z}{2}\right| < 1$$

$$X(z) = \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} 1 \, z^{n} - \sum_{n=0}^{\infty} 2^{-n-1} z^{n}$$

$$= \sum_{n=-1}^{-\infty} \left[-1 + 2^{n-1}\right] z^{-n}$$

③ D_2 $1 < |z| < 2$ two-sided



$$\left|\frac{1}{z}\right| < 1 \qquad \left|\frac{z}{2}\right| < 1$$

$$X(z) = \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$

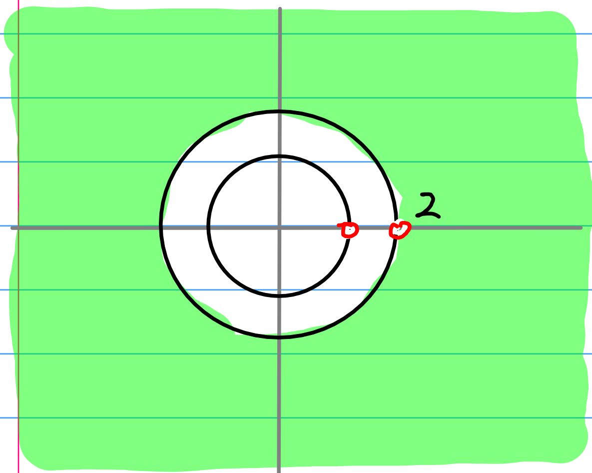
$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n-1} + \sum_{n=0}^{\infty} 2^{-n-1} z^{n}$$

$$= \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$

Z.T. first

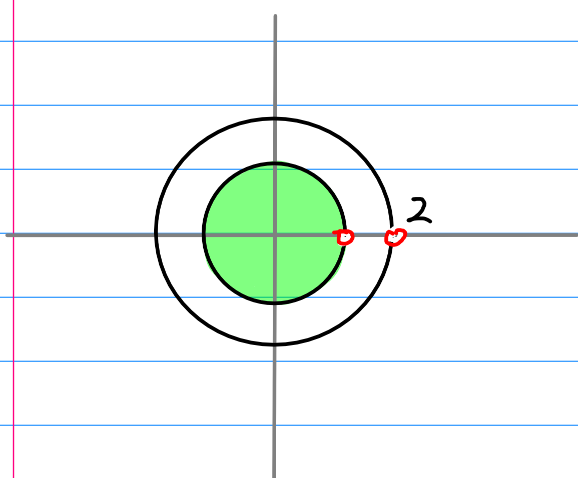
$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

① D_1 $|z| > 2$ causal $x_n = 0 \quad (n<0)$



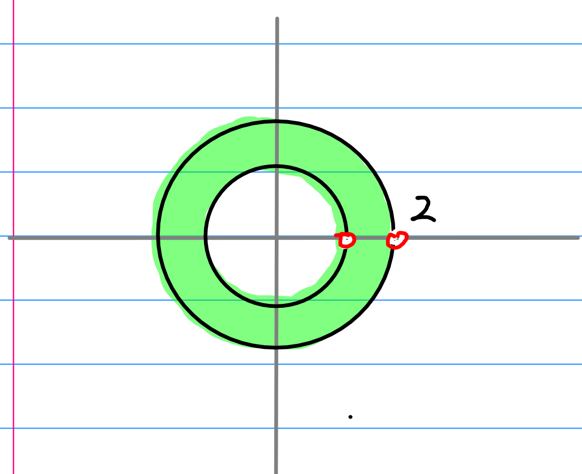
$$X(z) = \sum_{n=1}^{\infty} \left[1 - 2^{-n+1}\right] z^{-n}$$

② D_3 $|z| < 1$ anti-causal $x_n = 0 \quad (n>0)$



$$X(z) = \sum_{n=-1}^{\infty} \left[-1 + 2^{n-1}\right] z^{-n}$$

③ D_2 $1 < |z| < 2$ two-sided



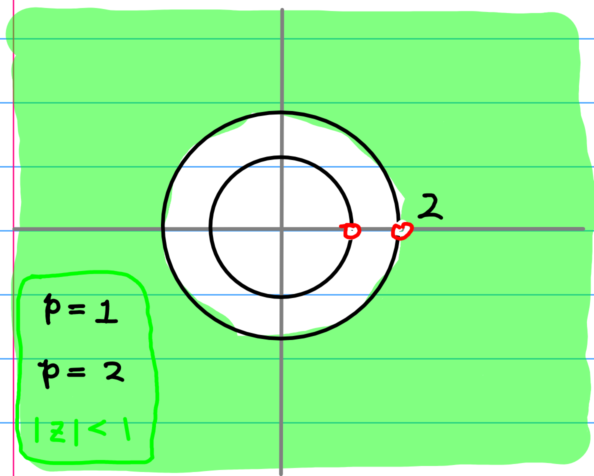
$$X(z) = \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$

$$X(z) = \frac{-1}{(z-1)(z-2)} \qquad f(z) =$$

Z.T. first

I − 1

I D_1 $|z| > 2$ $\left[\left|\frac{1}{z}\right| < 1 ,\ \left|\frac{2}{z}\right| < 1 \right]$

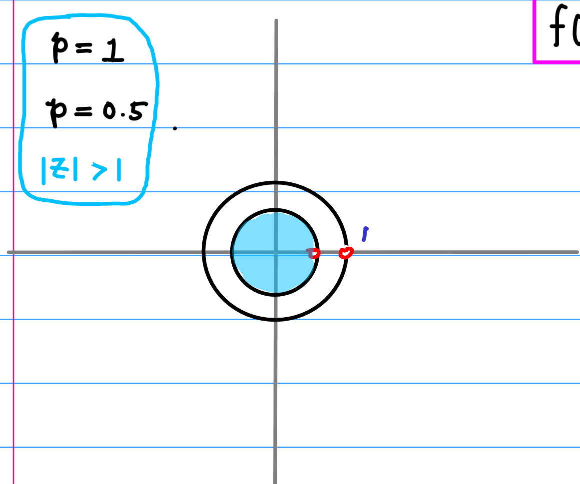


$$X(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} 1\, z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1}$$

$$= \sum_{n=1}^{\infty} \left[1 - 2^{n-1}\right] z^{-n}$$



$$\boxed{f(z) = X(z^{-1})} = \sum_{n=1}^{\infty} \left[1 - 2^{n-1}\right] z^{n}$$

$$= \sum_{n=1}^{\infty} 1 \cdot z^n - 2^{n-1} \cdot z^n$$

$$= \frac{z}{1-\left(\frac{z}{1}\right)} - \frac{z}{1-\left(\frac{2z}{1}\right)}$$

$$= -\frac{z}{z-1} + \frac{0.5z}{(z-0.5)}$$

$$= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} z$$

$$= \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = f(z)$$

Z.T. first

Ⅱ -1

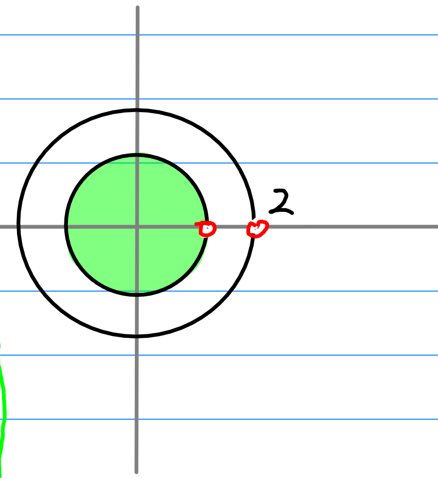
$$X(z) = \frac{-1}{(z-1)(z-2)} \qquad f(z) =$$

Ⅱ D_2 $|z| < 1$ $\left[\left|\frac{z}{1}\right| < 1, \quad \left|\frac{z}{2}\right| < 1 \right]$

$p = 1$

$p = 2$

$|z| > 2$



$$X(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} (-1)\left(\frac{z}{1}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^n$$

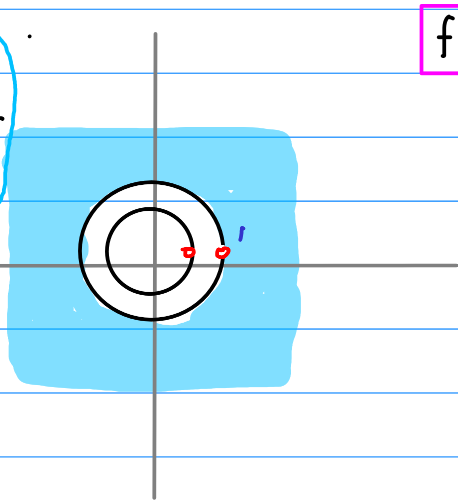
$$= -\sum_{n=0}^{\infty} 1 \cdot z^n + \sum_{n=0}^{\infty} 2^{-n-1} z^n$$

$$= \sum_{n=0}^{-\infty} \left[-1 + 2^{n-1}\right] z^{-n}$$

$p = 1$

$p = 0.5$

$|z| < \frac{1}{2}$



$$\boxed{f(z) = X(z^{-1})} = \sum_{n=0}^{-\infty} \left[-1 + 2^{n-1}\right] z^n$$

$$= \sum_{n=0}^{\infty} -1 \cdot z^{-n} + \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^{-n}$$

$$= -\frac{1}{1-\left(\frac{1}{z}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{1}{2z}\right)}$$

$$= -\frac{z}{z-1} + \frac{0.5z}{z-0.5}$$

$$= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} z$$

$$= \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$X(z) = \frac{-1}{(z-1)(z-2)}$$

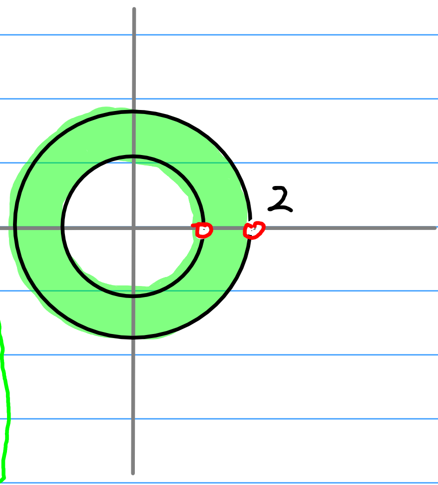
$$X(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = f(z)$$

$$X(z) = \frac{-1}{(z-1)(z-2)} \qquad f(z) =$$

Z.T. first

III -1

III D_3 $1 < |z| < 2$ $\left[\left|\frac{1}{z}\right| < 1 , \quad \left|\frac{z}{2}\right| < 1 \right]$



$p = 1$

$p = 2$

$1 < |z| < 2$

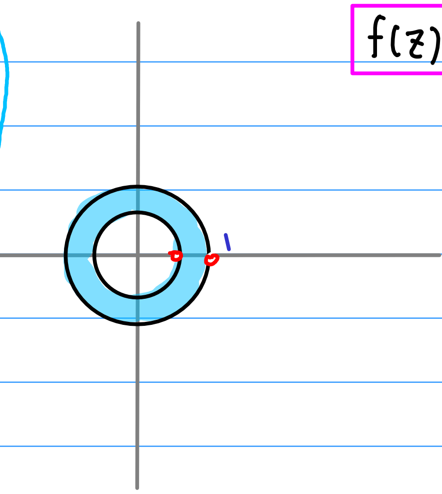
$$X(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} z^{-n-1} + \sum_{n=0}^{\infty} 2^{-n-1} z^n$$

$$= \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$



$p = 1$

$p = 0.5$

$\frac{1}{2} < |z| < 1$

$$\boxed{f(z) = X(z^{-1})} = \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^n$$

$$= \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^{-n}$$

$$= \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)}$$

$$= \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$

$$= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} z$$

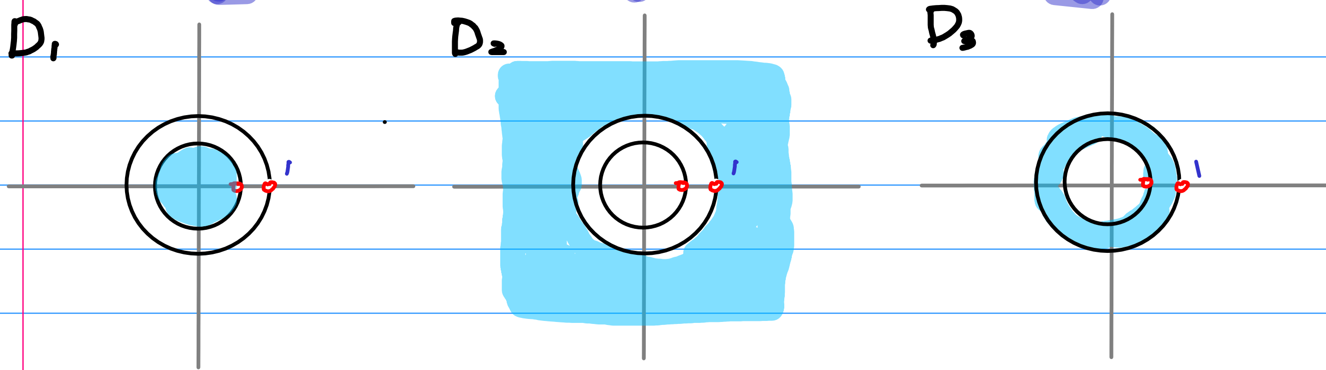
$$= \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = f(z)$$

L.S. first

$$\boxed{f(z) = \frac{-0.5\,z^2}{(z-1)(z-0.5)}}$$



D_1	D_2	D_3
a_n ?	a_n ?	a_n ?
$\Vert$	$\Vert$	$\Vert$
x_n	x_n	x_n
$X(z)$?	$X(z)$?	$X(z)$?

$$\boxed{X(z) = \,?}$$

$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$\frac{z}{(z-1)(z-0.5)} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$= \frac{2z-1-z+1}{(z-1)(z-0.5)}$$

$$\frac{-0.5z^2}{(z-1)(z-0.5)} = -0.5\left(\frac{2z}{z-1} - \frac{z}{z-0.5}\right) = \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$

$$\frac{-z}{z-1} = \begin{cases} \dfrac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} = \sum_{n=0}^{\infty} z \cdot z^n = \sum_{n=0}^{\infty} 1 \cdot z^{n+1} & \left|\frac{z}{1}\right| < 1 \\ \dfrac{-1}{1-\left(\frac{1}{z}\right)} = -\sum_{n=0}^{\infty} 1 \cdot z^{-n} = -\sum_{n=0}^{\infty} 1 \cdot z^{-n} & \left|\frac{1}{z}\right| < 1 \end{cases}$$

$$\frac{0.5z}{z-0.5} = \begin{cases} \dfrac{-\left(\frac{z}{1}\right)}{1-\left(\frac{2z}{1}\right)} = -\sum_{n=0}^{\infty} \left(\frac{z}{1}\right)\left(\frac{2z}{1}\right)^n = -\sum_{n=0}^{\infty} 2^n z^{n+1} & \left|\frac{2z}{1}\right| < 1 \\ \dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{1}{2z}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^{-n} & \left|\frac{1}{2z}\right| < 1 \end{cases}$$

$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$ $\quad X(z) = \frac{-1}{(z-1)(z-2)}$

L.S. first

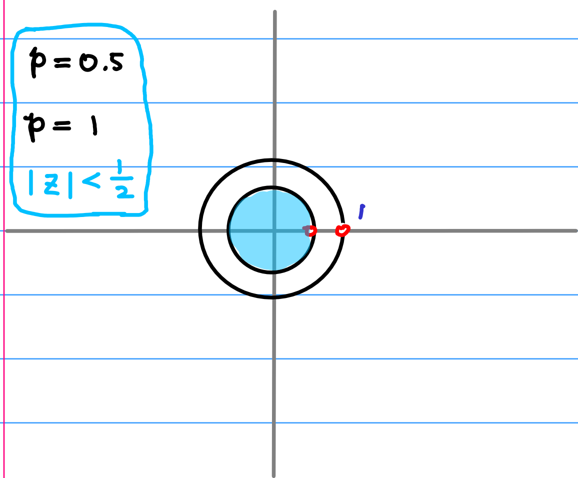
I − 1

I $\quad D_1 \quad |z| < 0.5 \quad \left[\, \left|\frac{z}{1}\right| < 1 ,\quad \left|\frac{2z}{1}\right| < 1 \,\right]$

$p = 0.5$

$p = 1$

$|z| < \frac{1}{2}$



$$f(z) = \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$

$$= \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} + \frac{-\left(\frac{z}{1}\right)}{1-\left(\frac{2z}{1}\right)}$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{n+1} - \sum_{n=0}^{\infty} 2^n z^{n+1}$$

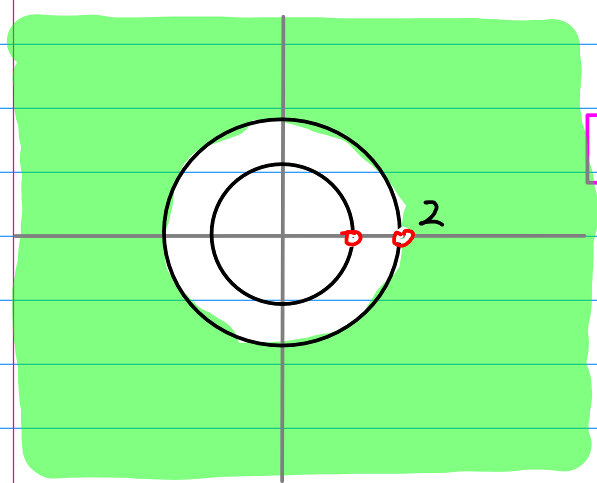
$$= \sum_{n=0}^{\infty} \left[1 - 2^n\right] z^{n+1}$$

$$= \sum_{n=1}^{\infty} \left[1 - 2^{n-1}\right] z^{n} \qquad |z| < \frac{1}{2}$$

$p =$

$p =$

$|z| > 1$



$$\boxed{X(z) = f(z^{-1})} = \sum_{n=1}^{\infty} \left[1 - 2^{n-1}\right] z^{-n} \qquad |z| > 2$$

$$= \sum_{n=1}^{\infty} 1 \cdot z^{-n} - \sum_{n=1}^{\infty} 2^{n-1} \cdot z^{-n}$$

$$= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$= \frac{1}{z-1} - \frac{1}{z-2}$$

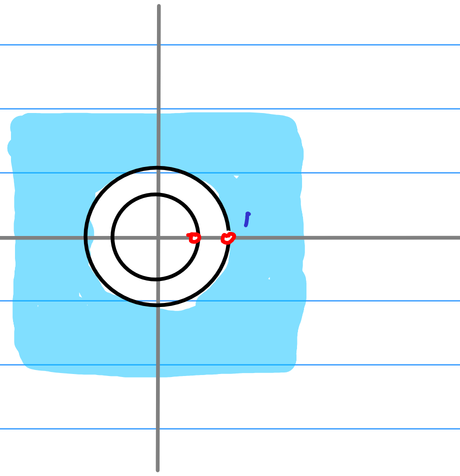
$$= \frac{z-2-z+1}{(z-1)(z-2)}$$

$$= \frac{-1}{(z-1)(z-2)}$$

$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$f(z^{-1}) = \frac{-0.5z^{-2}}{(z^{-1}-1)(z^{-1}-0.5)} = \frac{-0.5}{(1-z)(1-0.5z)} = \frac{-1}{(z-1)(z-2)} = X(z)$$

II D_2 $\quad 1<|z| \quad \left[\,\left|\frac{1}{z}\right|<1 \;,\quad \left|\frac{1}{2z}\right|<1\,\right]$



$p = 0.5$

$p = 1$

$|z| > \frac{1}{2}$

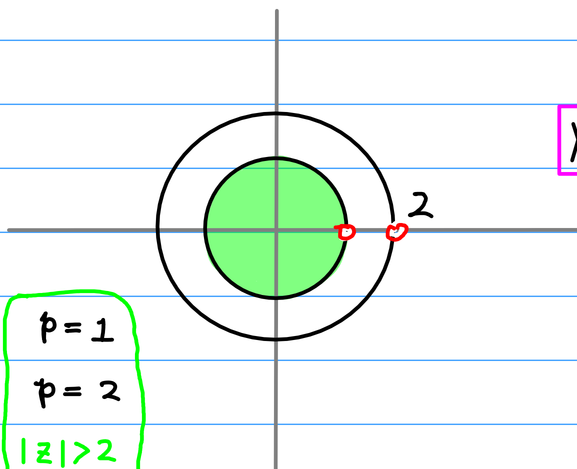
$$f(z) = \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$

$$= \frac{-1}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)}$$

$$= -\sum_{n=0}^{\infty} 1\cdot\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{1}{2z}\right)^n$$

$$= \sum_{n=0}^{\infty}\left[2^{-n-1}-1\right] z^{-n}$$

$$= \sum_{n=0}^{-\infty}\left[2^{n-1}-1\right] z^{n} \qquad |z|>1$$



$p = 1$

$p = 2$

$|z| > 2$

$$\boxed{X(z) = f(z^{-1})} = \sum_{n=0}^{-\infty}\left[2^{n-1}-1\right] z^{-n} \qquad |z|>2$$

$$= \sum_{n=0}^{\infty} 2^{-n-1}\cdot z^n - \sum_{n=0}^{\infty} 1\cdot z^n$$

$$= \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)} - \frac{1}{1-\left(\frac{z}{1}\right)}$$

$$= \frac{-1}{z-2} + \frac{1}{z-1}$$

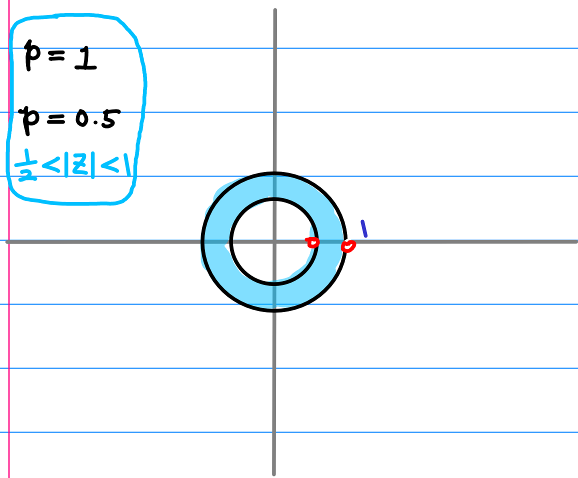
$$= \frac{-z+1+z-2}{(z-1)(z-2)}$$

$$= \frac{-1}{(z-1)(z-2)}$$

$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$f(z^{-1}) = \frac{-0.5z^{-2}}{(z^{-1}-1)(z^{-1}-0.5)} = \frac{-0.5}{(1-z)(1-0.5z)} = \frac{-1}{(z-1)(z-2)} = X(z)$$

$$\text{III}\quad D_3 \quad 0.5 < |z| < 1 \qquad \left[\ \left|\frac{z}{1}\right| < 1\ ,\quad \left|\frac{1}{2z}\right| < 1\ \right]$$



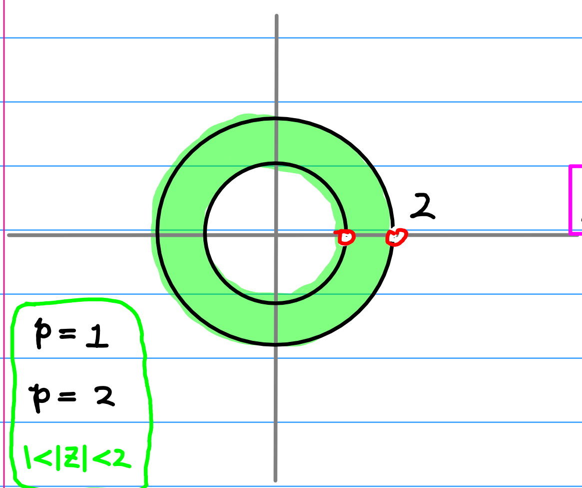
$$f(z) = \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$

$$= \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)}$$

$$= \sum_{n=0}^{\infty} z \cdot z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{1}{2z}\right)^n$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{n+1} + \sum_{n=0}^{\infty} 2^{-n-1}\, z^{-n}$$

$$= \sum_{n=1}^{\infty} 1 \cdot z^{n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{n} \qquad |z| < \frac{1}{2}$$



$$\boxed{X(z) = f(z^{-1})} = \sum_{n=1}^{\infty} 1 \cdot z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n} \qquad |z| > 2$$

$$= \sum_{n=1}^{\infty} 1 \cdot z^{-n} + \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^{n}$$

$$= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$$

$$= \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{z-2-z+1}{(z-1)(z-2)}$$

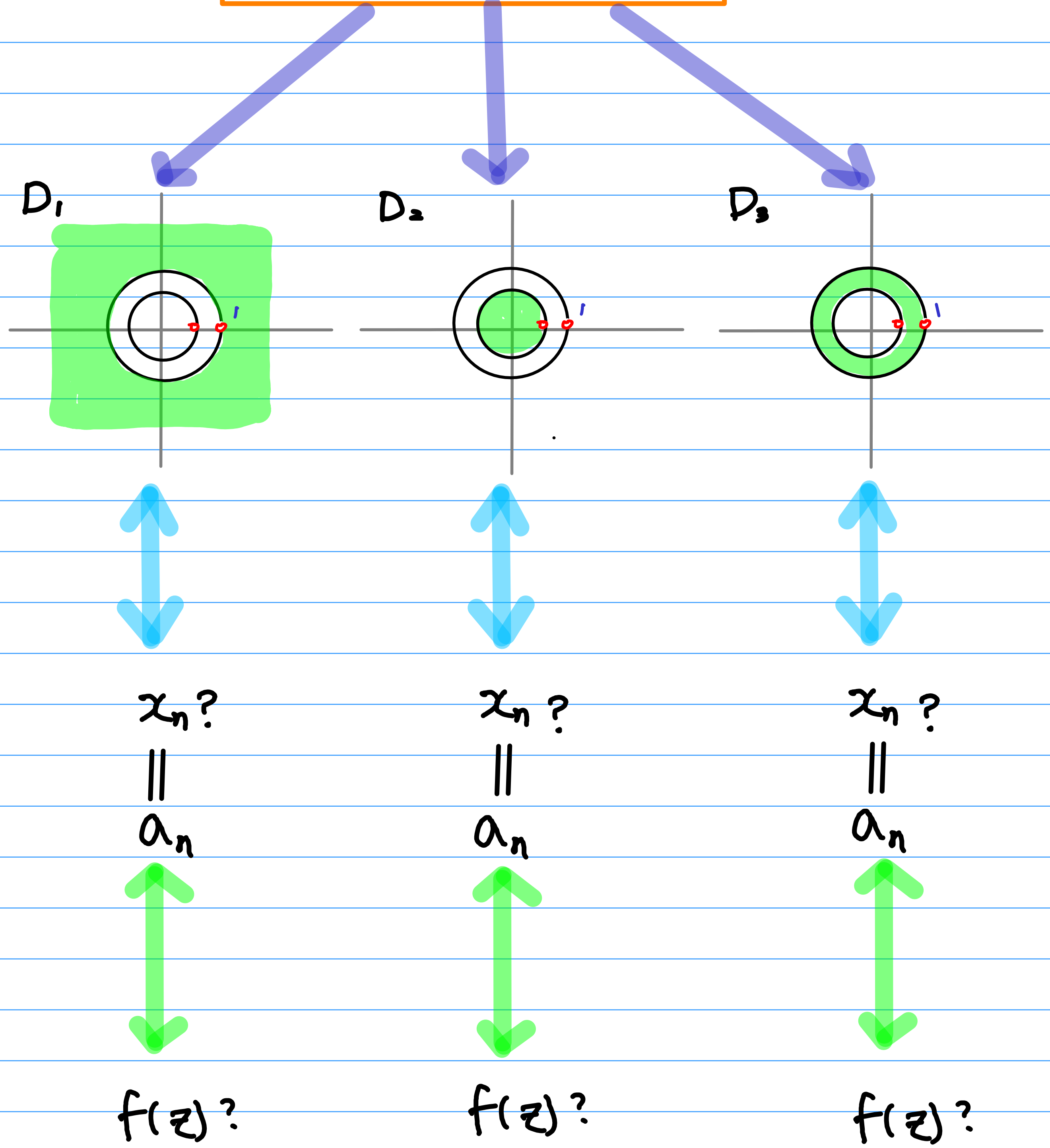
$$= \frac{-1}{(z-1)(z-2)}$$

$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$f(z^{-1}) = \frac{-0.5z^{-2}}{(z^{-1}-1)(z^{-1}-0.5)} = \frac{-0.5}{(1-z)(1-0.5z)} = \frac{-1}{(z-1)(z-2)} = X(z)$$

Z.T. first

$$\boxed{X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}}$$



Z.T. first

$$f(z) = \frac{-1}{(z-1)(z-2)} = X(z)$$

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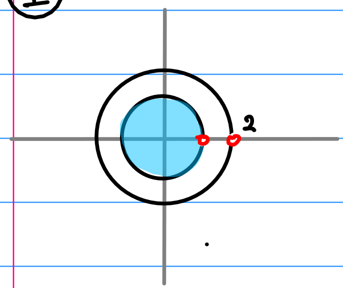

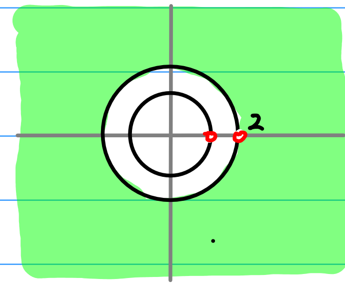

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

$$x_n = \begin{cases} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

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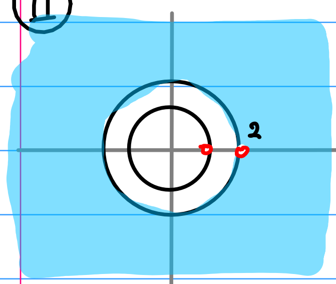

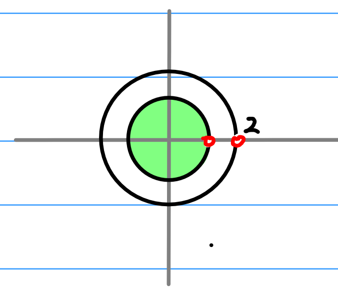

$$a_n = \begin{cases} 0 & (n \geq 0) \\ 1 - \left(\frac{1}{2}\right)^{n+1} & (n < 0) \end{cases}$$

$$x_n = \begin{cases} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{cases}$$

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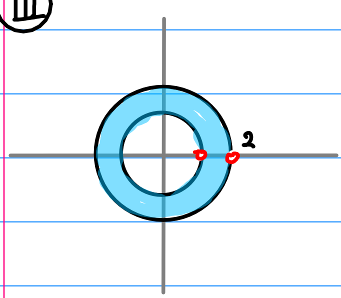
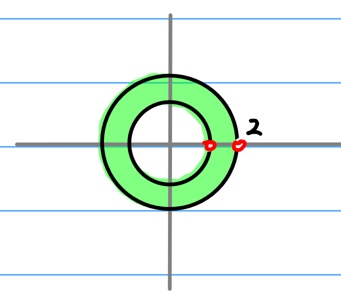




$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

$$x_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

$$x_n = \begin{cases} 0 & (n > 0) \\ \left(\frac{1}{2}\right)^{-n+1} - 1 & (n \leq 0) \end{cases}$$

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n - \sum_{n=0}^{\infty} 1 \cdot z^n$$

$$X(z) = \sum_{n=0}^{-\infty} \left(\frac{1}{2}\right)^{-n+1} z^{-n} - \sum_{n=0}^{-\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n - \sum_{n=0}^{\infty} 1 \cdot z^n$$

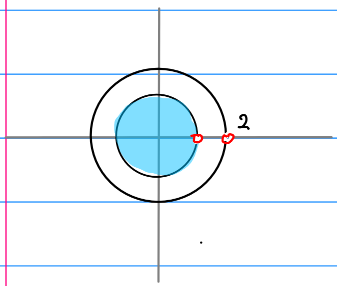
$$= \frac{\left(\frac{1}{2}\right)}{1 - \left(\frac{z}{2}\right)} - \frac{1}{1 - z}$$

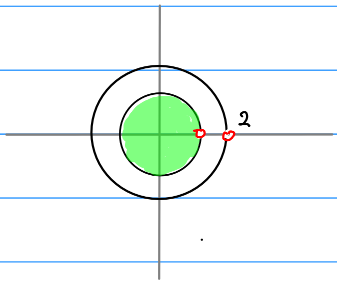
$$= \frac{-1}{z-2} + \frac{1}{z-1}$$

$$= \frac{-z+1+z-2}{(z-1)(z-2)}$$

$$= \frac{-1}{(z-1)(z-2)}$$

$$\left|\frac{z}{2}\right| < 1 \quad \left|\frac{z}{1}\right| < 1$$





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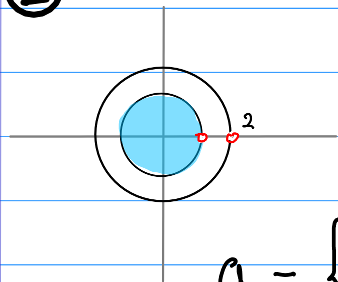
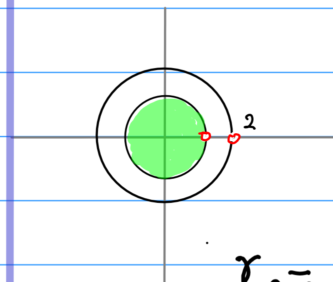


$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

$$f(z) = \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^n - \sum_{n=0}^{\infty} 1 \cdot z^n$$



$$x_n = \begin{cases} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{cases}$$

$$X(z) = \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n} - \sum_{n=0}^{-\infty} 1 \cdot z^{-n}$$

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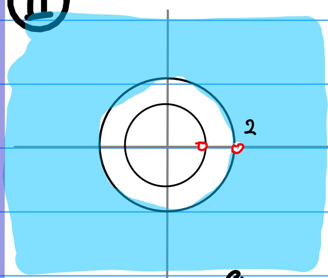
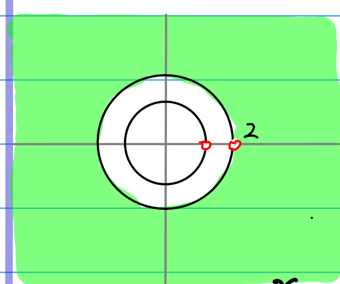


$$a_n = \begin{cases} 0 & (n \geq 0) \\ 1 - \left(\frac{1}{2}\right)^{n+1} & (n < 0) \end{cases}$$

$$f(z) = \sum_{n=-1}^{-\infty} 1 \cdot z^n - \sum_{n=-1}^{-\infty} 2^{-n-1} \cdot z^n$$



$$x_n = \begin{cases} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

$$X(z) = \sum_{n=1}^{\infty} 1 \cdot z^{-n} - \sum_{n=1}^{\infty} 2^{n-1} \cdot z^{-n}$$

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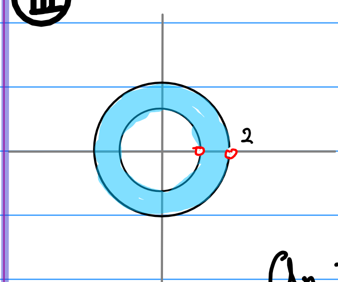

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

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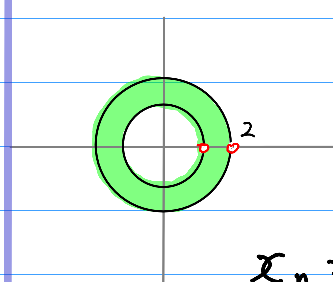
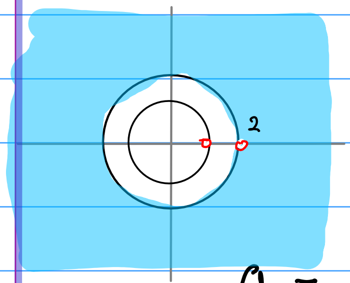


$$x_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$$X(z) = \sum_{n=1}^{\infty} 1 \cdot z^{-n} - \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$



$$a_n = \begin{cases} 0 & (n \geq 0) \\ 1-\left(\frac{1}{2}\right)^{n+1} & (n<0) \end{cases}$$

$$f(z) = \sum_{n=-1}^{-\infty} 1 \cdot z^n - \sum_{n=-1}^{-\infty} 2^{-n-1} z^n$$

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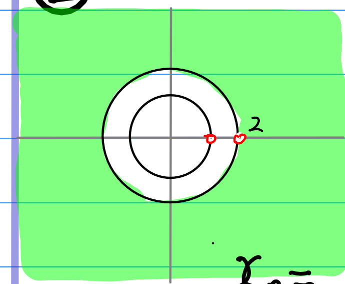
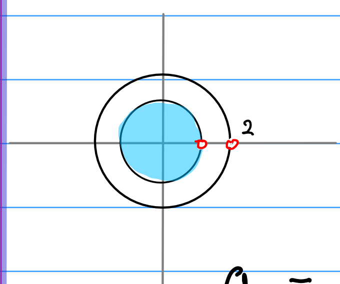


$$x_n = \begin{cases} 1-2^{n-1} & (n>0) \\ 0 & (n \leq 0) \end{cases}$$

$$X(z) = \sum_{n=1}^{\infty} 1 \cdot z^{-n} - \sum_{n=1}^{\infty} 2^{n-1} z^{-n}$$



$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1}-1 & (n \geq 0) \\ 0 & (n<0) \end{cases}$$

$$f(z) = \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^n - \sum_{n=0}^{\infty} 1 \cdot z^n$$

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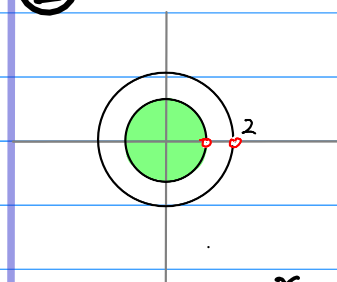
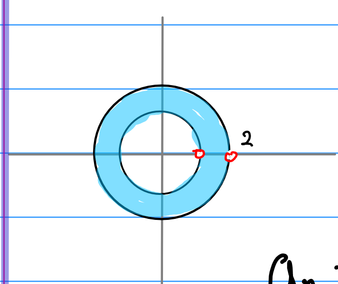


$$x_n = \begin{cases} 0 & (n>0) \\ 2^{n-1}-1 & (n \leq 0) \end{cases}$$

$$X(z) = \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n} - \sum_{n=0}^{-\infty} 1 \cdot z^{-n}$$



$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} & (n \geq 0) \\ 1 & (n<0) \end{cases}$$

$$f(z) = \sum_{n=-1}^{-\infty} 1 \cdot z^n + \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^n$$

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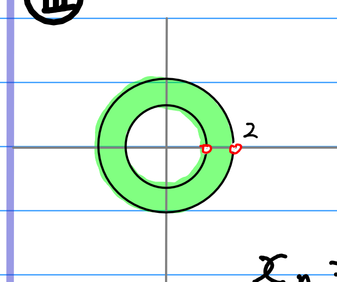

$$x_n = \begin{cases} 1 & (n>0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$$X(z) = \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n} + \sum_{n=1}^{\infty} 1 \cdot z^{-n}$$