Differentiation of Continuous Functions

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Oct 14, 2024

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Based on
Introduction to Matrix Algebra, Autar Kaw
https://ma.mathforcollege.com

Outline

Outline

Forward Difference Approximation (1)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

for a finite $\Delta x > 0$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Forward Difference Approximation (2)

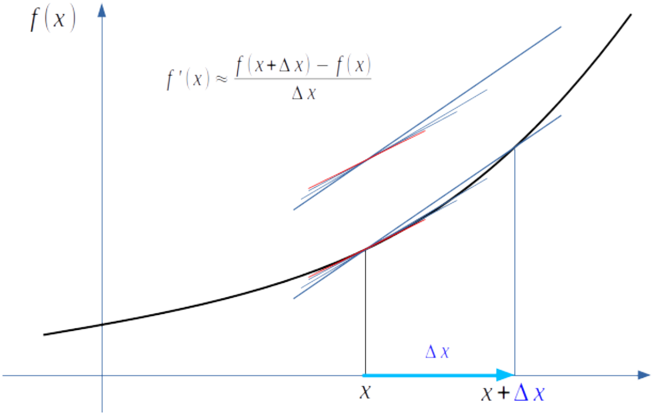

Figure: forward difference approximation

Forward Difference Approximation (3)

a forward difference approximation

as you are taking a point forward from x.

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx forwad as $x = x_{i+1}$.

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\begin{aligned} f'(x_i) &\approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x} \\ &= \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \end{aligned}$$

Outline

Backward Difference Approximation (1a)

forward difference approximation
for a finite $\Delta x > 0$,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

backward difference approximation
for a finite $\Delta x < 0$, then $-\Delta x > 0$,

$$f'(x) \approx \frac{f(x-\Delta x) - f(x)}{-\Delta x}$$
$$= \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Backward Difference Approximation (1b)

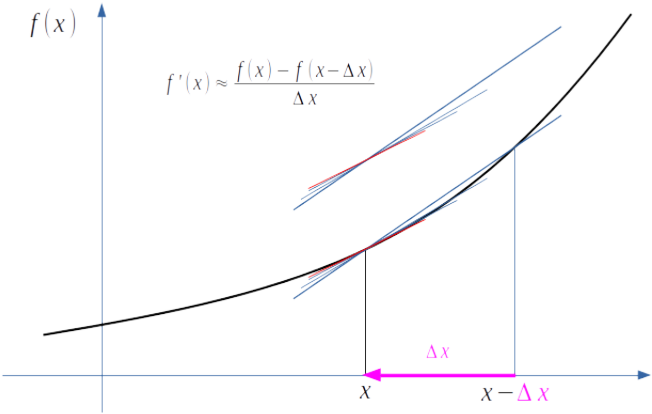

Figure: backward difference approximation (a)

Backward Difference Approximation (2a)

forward difference approximation
for a finite $\Delta x > 0$,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

backward difference approximation
for a finite $\Delta x > 0$, then $-\Delta x < 0$,

$$f'(x) \approx \frac{f(x) - f(x-\Delta x)}{x - (x-\Delta x)} = \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Backward Difference Approximation (2b)

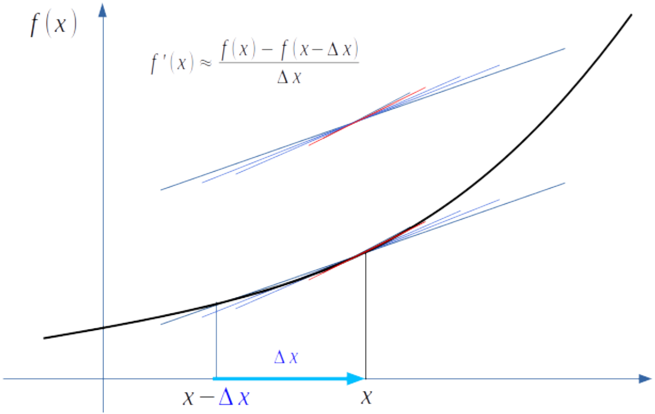

Figure: backward difference approximation (b)

Backward Difference Approximation (3)

a backward difference approximation

as you are taking a point backward from x.

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx backwad as $x = x_{i-1}$.

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$
$$= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

Outline

Taylor Series (1)

the Taylor series of a function $f(x)$,
that is infinitely differentiable at a point a is the power series

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$

Taylor Series (2)

If $f(x)$ is given by a convergent power series
in an open disk centred at a,
it is said to be *analytic* in this region.

Thus, for x in this region,
f is given by a convergent power series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Approximating the first derivative

A Taylor expansion approximates $f(x)$, using $f(a), f'(a), f''(a), \cdots$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- for forward difference approximatin

  $$x_i = a, \quad x_{i+1} = x, \quad \Delta x = x_{i+1} - x_i$$

- for backward difference approximatin

  $$x_i = a, \quad x_{i-1} = x, \quad \Delta x = x_i - x_{i-1}$$

Deriving Forward Difference Approximation (1)

A Taylor expansion approximates $f(x)$, using $f(a), f'(a), f''(a), \cdots$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Let $x_i = a$ and $x_{i+1} = x$
(from a toward x, approximate $f(x_{i+1})$, using information at x_i)

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \cdots$$

Deriving Forward Difference Approximation (2)

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots$$

$$f(x_{i+1}) = f(x_i) + \{f'(x_i)(\Delta x)\} + \frac{f''(x_i)}{2!}(\Delta x)^2 + \cdots$$

$$\underline{f(x_{i+1}) - f(x_i)} - \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots = \{f'(x_i)(\Delta x)\}$$

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x) - \cdots = f'(x_i)$$

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x) = f'(x_i)$$

Deriving Backward Difference Approximation (1)

A Taylor expansion approximates $f(x)$, using $f(a), f'(a), f''(a), \cdots$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Let $x_i = a$ and $x_{i-1} = x$
(from a toward x, approximate $f(x_{i-1})$, using information at x_i)

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \cdots$$

Substituting for convenience $\Delta x = x_i - x_{i-1}$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

Deriving Forward Difference Approximation (2)

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \cdots$$

$$f(x_{i-1}) = f(x_i) - \{f'(x_i)(\Delta x)\} + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

$$\{f'(x_i)(\Delta x)\} = \underline{f(x_i) - f(x_{i-1})} + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + \frac{f''(x_i)}{2!}(\Delta x) - \cdots$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + O(\Delta x)$$

Forward and Backward Difference Approximation

- for forward difference approximatin

  $x_i = a, \quad x_{i+1} = x, \quad \Delta x = x_{i+1} - x_i$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)$$

- for forward difference approximatin

  $x_i = a, \quad x_{i-1} = x, \quad \Delta x = x_i - x_{i-1}$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + O(\Delta x)$$

Approximation Errors

- the $O(\Delta x)$ term shows that the error in the approximation is of the order of Δx
- both forward and backward difference approximation of the first derivative are accurate in the order of $O(\Delta x)$
- to get better approximations the Central divided difference approximation of the first derivative.

Outline

Deriving Central Divide Approximation (1)

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Forward difference approximation : let $x_i = a$ and $x_{i+1} = x$
(from a toward x, approximate $f(x_{i+1})$, using information at x_i)

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots$$

Backward difference approximation : let $x_i = a$ and $x_{i-1} = x$
(from a toward x, approximate $f(x_{i-1})$, using information at x_i)

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \cdots$$

Deriving Central Divide Approximation (2)

Forward difference approximation :
substitute $\Delta x_1 = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x_1) + \frac{f''(x_i)}{2!}(\Delta x_1)^2 + \cdots$$

Backward difference approximation :
substitute $\Delta x_2 = x_i - x_{i-1}$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x_2) + \frac{f''(x_i)}{2!}(\Delta x_2)^2 - \cdots$$

Deriving Central Divide Approximation (3)

the same $\Delta x = \Delta x_1 = \Delta x_2$ is used

in forward and backward difference approximation

	backward		Δx		forward	
$i=1$	$f(x_0)$	$\leftarrow$	$x_1 - x_0$	$\rightarrow$	$f(x_1)$	$i=0$
$i=2$	$f(x_1)$	$\leftarrow$	$x_2 - x_1$	$\rightarrow$	$f(x_2)$	$i=1$
$i=3$	$f(x_2)$	$\leftarrow$	$x_3 - x_2$	$\rightarrow$	$f(x_3)$	$i=2$
$i=4$	$f(x_3)$	$\leftarrow$	$x_1 - x_3$	$\rightarrow$	$f(x_4)$	$i=3$
$i=5$	$f(x_4)$	$\leftarrow$	$x_1 - x_4$	$\rightarrow$	$f(x_5)$	$i=4$
$i=6$	$f(x_5)$	$\leftarrow$	$x_1 - x_5$	$\rightarrow$	$f(x_6)$	$i=5$
	$\vdots$		$\vdots$		$\vdots$	

Deriving Central Divide Approximation (4)

Forward difference approximation : substitute $\Delta x_1 = x_{i+1} - x_i$
Backward difference approximation : substitute $\Delta x_2 = x_i - x_{i-1}$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x_1) + \frac{f''(x_i)}{2!}(\Delta x_1)^2 + \cdots \quad (1)$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x_2) + \frac{f''(x_i)}{2!}(\Delta x_2)^2 - \cdots \quad (2)$$

Deriving Central Divide Approximation

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x_1) + \frac{f''(x_i)}{2!}(\Delta x_1)^2 + \cdots \quad (1)$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x_2) + \frac{f''(x_i)}{2!}(\Delta x_2)^2 - \cdots \quad (2)$$

subtracting eq(2) from eq(1)

$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)(\Delta x) + \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots$$

$$2f'(x_i)(\Delta x) = f(x_{i+1}) - f(x_{i-1}) - \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2(\Delta x)} - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^2 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O\left((\Delta x)^2\right)$$

Central Divided Approximation

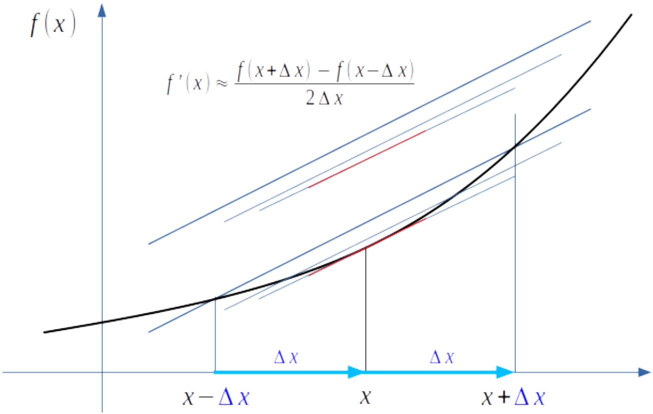

Figure: central difference approximation

Outline

Higher Order Derivatives (1)

Forward Difference Approximation:
Let $x_{i+1} = x_i + \Delta x$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots \quad (3)$$

Let $x_{i+2} = x_i + 2\Delta x$

$$f(x_{i+2}) = f(x_i) + f'(x_i)(2\Delta x) + \frac{f''(x_i)}{2!}(2\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(2\Delta x)^3 + \cdots \quad (4)$$

Higher Order Derivatives (2)

(4)

$$f(x_{i+2}) = f(x_i) + f'(x_i)(2\Delta x) + \frac{f''(x_i)}{2!}(2\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(2\Delta x)^3 + \cdots$$

$-2*(3)$

$$-2f(x_{i+1}) = -2f(x_i) - f'(x_i)2(\Delta x) - \frac{f''(x_i)}{2!}2(\Delta x)^2 - \frac{f^{(3)}(x_i)}{3!}2(\Delta x)^3 + \cdots$$

$(4) - 2*(3)$

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)(\Delta x)^2 + f^{(3)}(x_i)(\Delta x)^3 + \cdots$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} - f^{(3)}(x_i)(\Delta x)$$

Higher Order Derivatives (3)

Forward Difference Approximation:
Let $x_{i+1} = x_i + \Delta x$ and $x_{i+2} = x_i + 2\Delta x$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots \qquad (3)$$

$$f(x_{i+2}) = f(x_i) + f'(x_i)(2\Delta x) + \frac{f''(x_i)}{2!}(2\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(2\Delta x)^3 + \cdots \qquad (4)$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} - f^{(3)}(x_i)(\Delta x)$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2} + O(\Delta x)$$

Higher Order Derivatives (4)

The formula given by equation (5) is a forward difference approximation of the second derivative and has the error of the order of $O(\Delta x)$.
for a better accuracy
We can get the central difference approximation of the second derivative.
Let $x_{i+1} = x_i + \Delta x$ and $x_{i-1} = x_i - \Delta x$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots \quad (6)$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots \quad (7)$$

Higher Order Derivatives (5)

Adding equations (6) and (7), gives

$$(6)+(7)$$

$$f(x_{i+1})+f(x_{i-1}) = 2f(x_i)+f''(x_i)(\Delta x)^2+2\frac{f^{(4)}(x_i)}{4!}(\Delta x)^4+\cdots$$

$$f(x_{i+1})-2f(x_i)+f(x_{i-1}) = f''(x_i)(\Delta x)^2+2\frac{f^{(4)}(x_i)}{4!}(\Delta x)^4+\cdots$$

$$f(x_{i+1})-2f(x_i)+f(x_{i-1})-2\frac{f^{(4)}(x_i)}{4!}(\Delta x)^4-\cdots = f''(x_i)(\Delta x)^2$$

$$\frac{f(x_{i+1})-2f(x_i)+f(x_{i-1})}{(\Delta x)^2}+O\left((\Delta x)^2\right) = f''(x_i)$$

Higher Order Derivatives (6)

Let $x_{i+1} = x_i + \Delta x$ and $x_{i-1} = x_i - \Delta x$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots \quad (6)$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots \quad (7)$$

$$f(x_{i+1}) + f(x_{i-1}) = f(x_i) + f''(x_i)(\Delta x)^2 + 2\frac{f^{(4)}(x_i)}{4!}(\Delta x)^4 + \cdots$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} + O\left((\Delta x)^2\right)$$

Tangent Lines

- as $h \to 0$, $\mathrm{Q} \to \mathrm{P}$
  and the secant line $\to$ the tangent line
- the slope of the tangent line

$$m_{tangent} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$