

# Differentiation of Discrete Functions

Young W Lim

Nov 16, 2024

Copyright (c) 2024 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts.

This work is licensed under a Creative Commons  
"Attribution-NonCommercial-ShareAlike 3.0 Unported"  
license.



Based on  
Introduction to Matrix Algebra, Autar Kaw  
<https://ma.mathforcollege.com>

# Outline

- 1 Approximations of a first derivative
  - Direct Fit Polynomials

# Outline

- 1 Approximations of a first derivative
  - Direct Fit Polynomials

# Direct Fit Polynomials

given  $n+1$  data points

$$(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$$

one can fit a  $n^{\text{th}}$  order **polynomial** given by

$$P_n(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n$$

to find the **first derivative**

$$P'_n(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2x + \dots + (n-1)a_{n-1}x^{n-2} + na_nx^{n-1}$$

similarly other derivatives can be found

# Lagrange Polynomials (1)

given  $(n+1)$  data points

$$(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$$

one can fit a  $n^{th}$  order **Lagrange polynomial** given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where  $n$  in  $f_n(x)$  stands for the  $n^{th}$  order polynomial that approximates the function  $y = f(x)$

given at  $(n+1)$  data points

as  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$

# Lagrange Polynomials (2)

one can fit a  $n^{th}$  order Lagrange polynomial given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where  $n$  in  $f_n(x)$  stands for the  $n^{th}$  order **polynomial** for the function  $y = f(x)$

$$f_n(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n = \sum_{i=0}^n L_i(x) f(x_i)$$

given at  $(n+1)$  data points as  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , and

$$L_i(x) = \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j}$$

# Lagrange Polynomials (3)

given a set of  $(n+1)$  nodes  $\{x_0, x_1, \dots, x_n\}$ ,  
 which must all be distinct,  $x_j \neq x_i$  for indices  $j \neq i$ ,

the **Lagrange basis** for polynomials of degree  $\leq n$  for those nodes  
 is the set of polynomials  $\{L_0(x), L_1(x), \dots, L_n(x)\}$

$$\begin{aligned}
 L_i(x) &= \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j} \\
 &= \frac{x - x_0}{x_i - x_0} \cdot \frac{x - x_1}{x_i - x_1} \cdot \dots \cdot \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot 1 \cdot \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdot \frac{x - x_{i+2}}{x_i - x_{i+2}} \cdot \dots \cdot \frac{x - x_n}{x_i - x_n}
 \end{aligned}$$

[https://en.wikipedia.org/wiki/Lagrange\\_polynomial](https://en.wikipedia.org/wiki/Lagrange_polynomial)



# Lagrange Polynomials (4)

each **Lagrange basis** of degree  $n$  take values

$L_i(x_j) = 0$  if  $j \neq i$  and  $L_i(x_i) = 1$ .

Using the **Kronecker delta** this can be written  $L_i(x_j) = \delta_{ij}$ .

Each **basis polynomial** can be explicitly described by the product:

$$\begin{aligned}
 L_i(x) &= \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j} \\
 &= \frac{x - x_0}{x_i - x_0} \cdot \frac{x - x_1}{x_i - x_1} \cdots \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot 1 \cdot \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdot \frac{x - x_{i+2}}{x_i - x_{i+2}} \cdots \frac{x - x_n}{x_i - x_n}
 \end{aligned}$$

[https://en.wikipedia.org/wiki/Lagrange\\_polynomial](https://en.wikipedia.org/wiki/Lagrange_polynomial)

# Lagrange Polynomials (5)

$$\begin{aligned}
 L_i(x_j) &= \prod_{k=0, k \neq i}^n \frac{x_j - x_k}{x_i - x_k} \\
 &= \frac{x_j - x_0}{x_i - x_0} \cdot \frac{x_j - x_1}{x_i - x_1} \cdots \frac{x_j - x_{i-1}}{x_i - x_{i-1}} \cdot 1 \cdot \frac{x_j - x_{i+1}}{x_i - x_{i+1}} \cdot \frac{x_j - x_{i+2}}{x_i - x_{i+2}} \cdots \frac{x_j - x_n}{x_i - x_n} \\
 &= 0 \quad \because \exists j = k \neq i \quad x_j = x_k
 \end{aligned}$$

$$\begin{aligned}
 L_i(x_i) &= \prod_{k=0, k \neq i}^n \frac{x_i - x_k}{x_i - x_k} \\
 &= \frac{x_i - x_0}{x_i - x_0} \cdot \frac{x_i - x_1}{x_i - x_1} \cdots \frac{x_i - x_{i-1}}{x_i - x_{i-1}} \cdot 1 \cdot \frac{x_i - x_{i+1}}{x_i - x_{i+1}} \cdot \frac{x_i - x_{i+2}}{x_i - x_{i+2}} \cdots \frac{x_i - x_n}{x_i - x_n} \\
 &= 1
 \end{aligned}$$

[https://en.wikipedia.org/wiki/Lagrange\\_polynomial](https://en.wikipedia.org/wiki/Lagrange_polynomial)

# Lagrange Polynomials (6)

hen to find the first derivative, one can differentiate  $f_n(x)$  for other derivatives.

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

For example, the second order Lagrange polynomial passing through  $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$$f_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$L_0(x) = \prod_{j=0, j \neq i}^2 \frac{x - x_j}{x_0 - x_j}, \quad L_1(x) = \prod_{j=0, j \neq i}^2 \frac{x - x_j}{x_1 - x_j}, \quad L_2(x) = \prod_{j=0, j \neq i}^2 \frac{x - x_j}{x_2 - x_j}$$

# Lagrange Polynomials (7)

the second order Lagrange polynomial passing through  $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$$f_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$L_0(x) = \prod_{j=0, j \neq i}^2 \frac{x - x_j}{x_0 - x_j}, \quad L_1(x) = \prod_{j=0, j \neq i}^2 \frac{x - x_j}{x_1 - x_j}, \quad L_2(x) = \prod_{j=0, j \neq i}^2 \frac{x - x_j}{x_2 - x_j}$$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2}, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2}, \quad L_2(x) = \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1}$$

$$f_2(x) = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} f(x_0) + \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} f(x_2)$$

# Lagrange Polynomials (8)

the second order Lagrange polynomial passing through  $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$$f_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$f_2(x) = \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2} f(x_0) + \frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2} f(x_1) + \frac{x-x_0}{x_2-x_0} \cdot \frac{x-x_1}{x_2-x_1} f(x_2)$$

$$\frac{d}{dx}(x-x_1)(x-x_2) = (x-x_2) + (x-x_1) = 2x - (x_1 + x_2)$$

$$\frac{d}{dx} f_2(x) = \frac{2x - (x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - (x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2x - (x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

$$\frac{d^2}{dx^2} f_2(x) = \frac{2f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{2f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{2f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

# Tangent Lines

- as  $h \rightarrow 0$ ,  $Q \rightarrow P$   
and the **secant line**  $\rightarrow$  the **tangent line**
- the slope of the **tangent line**

$$\begin{aligned}m_{\text{tangent}} &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{(a+h) - a} \\&= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}\end{aligned}$$



