Bisection Method

Young W Lim

Nov 30, 2024

Copyright (c) 2024 Young W. Lim.
Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts.

This work is licensed under a Creative Commons "Attribution-NonCommercial-ShareAlike 3.0 Unported" license.

Based on
Introduction to Matrix Algebra, Autar Kaw
https://ma.mathforcollege.com

Outline

Outline

Basis of Bisection Method (1)

An equation $f(x) = 0$, where $f(x)$ is a real continuous function, has at least one root between x_l and x_u if $f(x_l)f(x_u) < 0$.

At least one root exists between the two points
if the function is real, continuous, and changes sign.

Basis of Bisection Method (2)

If function does not change sign between two points,
roots of the equation may still exist between the two points.

Basis of Bisection Method (3)

If the function does not change sign between two points,
there may not be any roots for the equation between the two points.

Basis of Bisection Method (4)

If the function changes sign between two points,
more than one root for the equation may exist between the two points.

Algorithm for Bisection Method (1)

Step **1** Choose x_l and x_u as two guesses for the root such that $f(x_l)f(x_u) < 0$,
or in other words, $f(x)$ changes sign between x_l and x_u.
This was demonstrated in Figure 1

Algorithm for Bisection Method (2)

Step **2** Estimate the root, x_m of the equation $f(x) = 0$ as the mid point between x_l and x_u as

$$x_m = \frac{x_l + x_u}{2}$$

Algorithm for Bisection Method (3)

Step **3** check the followings

a) If $f(x_l) \cdot f(x_m) < 0$, then the root lies between x_l and x_m ;
then $x_l = x_l$; $x_u = x_m$.

b) If $f(x_l) \cdot f(x_m) > 0$, then the root lies between x_m and x_u;
then $x_l = x_m$; $x_u = x_u$.

c) If $f(x_l) \cdot f(x_m) = 0$; then the root is x_m.
Stop the algorithm if this is true.

Algorithm for Bisection Method (4)

Step **4** Find the new estimate of the root

$$x_m = \frac{x_l + x_u}{2}$$

find the absolute relative approximate error

$$|\varepsilon_a| = \left| \frac{x_m^{new} + x_m^{old}}{x_m^{new}} \right| \times 100$$

where x_m^{old} is previous estimate of root
and x_m^{new} is current estimate of root

Algorithm for Bisection Method (5)

Step **5** Compare the absolute relative approximate error $|\varepsilon_a|$ with the pre-specified error toleranceε_s
if $|\varepsilon_a| > \varepsilon_s$ then go to **Step 2** using new upper and lower guesses
else stop the algorithm

Note one should also check whether the number of iterations
is more than the maximum number of iterations allowed.
If so, one needs to terminate the algorithm
and notify the user about it

Advantages

- always convergent
- The root bracket gets halved with each iteration - guaranteed

Drawbacks (1)

- Slow convergence
- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (2)

- If a function $f(x)$ is such that it just touches the x-axis it will be unable to find the lower and upper guesses
- for example $f(x) = x^2$

Drawbacks (3)

- Function changes sign but root does not exist
- for example

$$f(x) = \frac{1}{x}$$

Bisection Method (1)

- the bisection method is a root-finding method
  that applies to any continuous function
  for which one knows two values with opposite signs.
- The method consists of repeatedly bisecting
  the interval defined by these values
  and then selecting the subinterval
  in which the function changes sign,
  and therefore must contain a root.

https://en.wikipedia.org/wiki/Bisection_method

Bisection Method (2)

- It is a very simple and robust method, but it is also relatively slow.
- Because of this, it is often used to obtain a rough approximation to a solution which is then used as a starting point for more rapidly converging methods.
- The method is also called the interval halving method, the binary search method, or the dichotomy method.

https://en.wikipedia.org/wiki/Bisection_method

Intermediate value theorem (1)

- the intermediate value theorem states that

  if f is a continuous function
  whose domain contains the interval $[a, b]$,
  then it takes on any given value between $f(a)$ and $f(b)$
  at some point within the interval.

  https://en.wikipedia.org/wiki/Intermediate_value_theorem

Intermediate value theorem (2)

This has two important corollaries:

- If a continuous function has values of opposite sign inside an interval, then it has a root in that interval (Bolzano's theorem).
- The image of a continuous function over an interval is itself an interval.

https://en.wikipedia.org/wiki/Intermediate_value_theorem