Bisection Method

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Based on
Introduction to Matrix Algebra, Autar Kaw
https://ma.mathforcollege.com

Outline

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1. Newton-Raphson Method
   - Newton-Raphson Method

Basis of Newton-Raphson Method (1)

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

.

Basis of Newton-Raphson Method (2)

$$\tan(\alpha) = \frac{\Delta y}{\Delta x}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_i - x_{i+1} = \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

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Algorithm for Newton-Raphson Method (1)

Step1 Evaluate $f'(x)$ symbollically

Step2 Use an initial guess of the root, x_i,
to estimate the new value of the root, x_{i+1}, as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Step3 Find the absolute relative approximate error $|\varepsilon_a|$as

$$|\varepsilon_a| = \left| \frac{x_{i+1} - x_i}{x_i} \right| \times 100$$

Algorithm for Newton-Raphson Method (2)

Step4 Compare the absolute relative approximate error $|\varepsilon_a|$
with the pre-specified relative error tolerance ε_s.

if $|\varepsilon_a| > \varepsilon_s$ then go to **Step 2** using new estimate of the root.
else stop the algorithm Also, check if the number of iterations

has exceeded
the maximum number of iterations allowed.
If so, one needs to terminate the algorithm and notify the user.

Advantages

- Converges fast (quadratic convergence), if it converges.
- Requires only one guess

Drawbacks (1)

- Divergence at inflection points
  Selection of the initial guess or an iteration value of the root
  that is close to the inflection point of the function
  may start diverging away from the root
  in ther Newton-Raphson method.

Drawbacks (2)

For example, to find the root of the equation $f(x) = (x-1)^3 + 0.512 = 0$.
The Newton-Raphson method reduces to $x_{i+1} = x_i - \frac{(x_i^3-1)^{3+0.512}}{3(x_i-1)^2}$.
Table 1 shows the iterated values of the root of the equation.
The root starts to diverge at Iteration 6
because the previous estimate of 0.92589
is close to the inflection point of $x = 1$.
Eventually after 12 more iterations the root converges
to the exact value of $x = 0.2$

Drawbacks (3)

Oscillations near local maximum and minimum
Results obtained from the Newton-Raphson method
may oscillate about the local maximum or minimum
without converging on a root
but converging on the local maximum or minimum.

Eventually, it may lead to division by a number close to zero
and may diverge.
For example for the equation has no real roots.

Drawbacks (4)

Root Jumping In some cases where the function
is oscillating and has a number of roots,
one may choose an initial guess close to a root.
However, the guesses may jump and converge to some other root.

Slow convergence (1)

The function f(x) = x2 has a root at 0.[13] Since f is continuously differentiable at its root, the theory guarantees that Newton's method as initialized sufficiently close to the root will converge. However, since the derivative f ′ is zero at the root, quadratic convergence is not ensured by the theory. In this particular example, the Newton iteration is given by
x n + 1 = x n − f (x n) f ′ (x n) = 1 2 x n . {\displaystyle x_{n+1}=x_{n}-{\frac {f(x_{n})}{f'(x_{n})}}={\frac {1}{2}}x_{n}.}
It is visible from this that Newton's method could be initialized anywhere and converge to zero, but at only a linear rate. If initialized at 1, dozens of iterations would be required before ten digits of accuracy are achieved.

https://en.wikipedia.org/wiki/Newton%27s_method

Slow convergence (2)

The function f(x) = x + x4/3 also has a root at 0, where it is continuously differentiable. Although the first derivative f ′ is nonzero at the root, the second derivative f ′′ is nonexistent there, so that quadratic convergence cannot be guaranteed. In fact the Newton iteration is given by
x n + 1 = x n − f (x n) f ′ (x n) = x 4 / 3 3 + 4 x 1 / 3 . {\displaystyle x_{n+1}=x_{n}-{\frac {f(x_{n})}{f'(x_{n})}}={\frac {x^{4/3}}{3+4x^{1/3}}}.}
From this, it can be seen that the rate of convergence is superlinear but subquadratic. This can be seen in the following tables, the left of which shows Newton's method applied to the above f(x) = x + x4/3 and the right of which shows Newton's method applied to f(x) = x + x2. The quadratic convergence in iteration shown on the right is illustrated by the orders of magnitude in the distance from the iterate to the true root (0,1,2,3,5,10,20,39,...) being approximately doubled from row to row. While the convergence on the left is superlinear, the order of magnitude is only multiplied by about 4/3 from row to row (0,1,2,4,5,7,10,13,...).

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Slow convergence (3)

The rate of convergence is distinguished from the number of iterations required to reach a given accuracy. For example, the function f(x) = x20 − 1 has a root at 1. Since f ′(1) ≠ 0 and f is smooth, it is known that any Newton iteration convergent to 1 will converge quadratically. However, if initialized at 0.5, the first few iterates of Newton's method are approximately 26214, 24904, 23658, 22476, decreasing slowly, with only the 200th iterate being 1.0371. The following iterates are 1.0103, 1.00093, 1.0000082, and 1.00000000065, illustrating quadratic convergence. This highlights that quadratic convergence of a Newton iteration does not mean that only few iterates are required; this only applies once the sequence of iterates is sufficiently close to the root.[14]

https://en.wikipedia.org/wiki/Newton%27s_method

depending on initialization (1)

The function f(x) = x(1 + x2)−1/2 has a root at 0. The Newton iteration is given by
x n + 1 = x n − f (x n) f ′ (x n) = − x n 3 . {\displaystyle x_{n+1}=x_{n}-{\frac {f(x_{n})}{f'(x_{n})}}=-x_{n}^{3}.}
From this, it can be seen that there are three possible phenomena for a Newton iteration. If initialized strictly between ± 1, the Newton iteration will converge (super-)quadratically to 0; if initialized exactly at 1 or -1, the Newton iteration will oscillate endlessly between ± 1; if initialized anywhere else, the Newton iteration will diverge.[15] This same trichotomy occurs for f(x) = arctan x.[13]

https://en.wikipedia.org/wiki/Newton%27s_method

depending on initialization (2)

In cases where the function in question has multiple roots, it can be difficult to control, via choice of initialization, which root (if any) is identified by Newton's method. For example, the function f(x) = x(x2 − 1)(x − 3)e−(x − 1)2/2 has roots at −1, 0, 1, and 3.[16] If initialized at −1.488, the Newton iteration converges to 0; if initialized at −1.487, it diverges to ∞; if initialized at −1.486, it converges to −1; if initialized at −1.485, it diverges to −∞; if initialized at −1.4843, it converges to 3; if initialized at −1.484, it converges to 1. This kind of subtle dependence on initialization is not uncommon; it is frequently studied in the complex plane in the form of the Newton fractal.

https://en.wikipedia.org/wiki/Newton%27s_method

Divergence (1)

Consider the problem of finding a root of f(x) = x1/3. The Newton iteration is x n + 1 = x n − f (x n) f ′ (x n) = x n − x n 1 / 3 1 3 x n − 2 / 3 = − 2 x n . {\displaystyle x_{n+1}=x_{n}-{\frac {f(x_{n})}{f'(x_{n})}}=x_{n}-{\frac {x_{n}^{1/3}}{{\frac {1}{3}}x_{n}^{-2/3}}}=-2x_{n}.}
Unless Newton's method is initialized at the exact root 0, it is seen that the sequence of iterates will fail to converge. For example, even if initialized at the reasonably accurate guess of 0.001, the first several iterates are −0.002, 0.004, −0.008, 0.016, reaching 1048.58, −2097.15, ... by the 20th iterate. This failure of convergence is not contradicted by the analytic theory, since in this case f is not differentiable at its root.
In the above example, failure of convergence is reflected by the failure of f(xn) to get closer to zero as n increases, as well as by the fact that successive iterates are growing further and further apart. However, the function f(x) = x1/3e−x2 also has a root at 0. The Newton iteration is given by
x n + 1 = x n − f (x n) f ′ (x n) = x n (1 − 3 1 − 6 x n 2) . {\displaystyle x_{n+1}=x_{n}-{\frac {f(x_{n})}{f'(x_{n})}}=x_{n}\left(1-{\frac {3}{1-6x_{n}^{2}}}\right).}

https://en.wikipedia.org/wiki/Newton%27s_method

Divergence (2)

n this example, where again f is not differentiable at the root, any Newton iteration not starting exactly at the root will diverge, but with both xn + 1 − xn and f(xn) converging to zero.[17] This is seen in the following table showing the iterates with initialization 1:
Although the convergence of xn + 1 − xn in this case is not very rapid, it can be proved from the iteration formula. This example highlights the possibility that a stopping criterion for Newton's method based only on the smallness of xn + 1 − xn and f(xn) might falsely identify a root.

https://en.wikipedia.org/wiki/Newton%27s_method

Oscillatory behavior (1)

t is easy to find situations for which Newton's method oscillates endlessly between two distinct values. For example, for Newton's method as applied to a function f to oscillate between 0 and 1, it is only necessary that the tangent line to f at 0 intersects the x-axis at 1 and that the tangent line to f at 1 intersects the x-axis at 0.[17] This is the case, for example, if $f(x) = x3 - 2x + 2$. For this function, it is even the case that Newton's iteration as initialized sufficiently close to 0 or 1 will asymptotically oscillate between these values. For example, Newton's method as initialized at 0.99 yields iterates 0.99, −0.06317, 1.00628, 0.03651, 1.00196, 0.01162, 1.00020, 0.00120, 1.000002, and so on. This behavior is present despite the presence of a root of f approximately equal to −1.76929.

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Undefinedness (1)

In some cases, it is not even possible to perform the Newton iteration. For example, if f(x) = x2 − 1, then the Newton iteration is defined by
x n + 1 = x n − f (x n) f ′ (x n) = x n 2 − 1 2 x n . {\displaystyle x_{n+1}=x_{n}-{\frac {f(x_{n})}{f'(x_{n})}}={\frac {x_{n}^{2}-1}{2x_{n}}}.}
So Newton's method cannot be initialized at 0, since this would make x1 undefined. Geometrically, this is because the tangent line to f at 0 is horizontal (i.e. f ′(0) = 0), never intersecting the x-axis.
Even if the initialization is selected so that the Newton iteration can begin, the same phenomenon can block the iteration from being indefinitely continued.
If f has an incomplete domain, it is possible for Newton's method to send the iterates outside of the domain, so that it is impossible to continue the iteration.[17] For example, the natural logarithm function f(x) = ln x has a root at 1, and is defined only for positive x. Newton's iteration in this case is given by
x n + 1 = x n − f (x n) f ′ (x n) = x n (1 − ln x n) . {\displaystyle x_{n+1}=x_{n}-{\frac {f(x_{n})}{f'(x_{n})}}=x_{n}(1-\ln x_{n}).}

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Undefinedness (2)

So if the iteration is initialized at e, the next iterate is 0; if the iteration is initialized at a value larger than e, then the next iterate is negative. In either case, the method cannot be continued.

https://en.wikipedia.org/wiki/Newton%27s_method