Reduced Row Echelon Form

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Based on
A First Course in Linear Algebra, R. A. Beezer
http://linear.ups.edu/fcla/front-matter.html

Outline

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Matrix (1)

Matrix

An $m \times n$ matrix is

a rectangular layout of numbers from C
having m rows and n columns.

- Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top)
- columns will be referenced starting from the left (i.e. column 1 is at the left).

Matrix (2)

Matrix

- use upper-case Latin letters from the start of the alphabet (A,B,C,. . .) to denote matrices and
- squared-off brackets to delimit the layout.
- For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

Column Vector

Column Vector

A column vector of size m is an ordered list of m numbers,

which is written in order vertically, starting at the top
and proceeding to the bottom.

- Column vectors will be written in bold,

  usually with lower case Latin letter
  from the end of the alphabet such as u, v, w, x, y, z.
- Some books like to write vectors with arrows, such as $\vec{u}$.
- Writing by hand, some like to put arrows on top of the symbol,
  or a tilde underneath the symbol, as in $\tilde{v}$.
- To refer to the entry or component of vector v in location i of the list,
  we write $[v]_i$.

Zero Column Vector

Zero Column Vector

The zero vector of size m is the column vector of size m
where each entry is the number zero,

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or defined much more compactly, [0]i=0 for $1 \leq i \leq m$.

Coefficient Matrix

Coefficient Matrix

For a system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the coefficient matrix is the m×n matrix

$$\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
 & & & & \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}$$

Vector of Constants

Vector of Constants

For a system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the vector of constants is the column vector of size m

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \\ b_m \end{bmatrix}$$

Solution Vector

Solution Vector

For a system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the solution vector is the column vector of size n

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ x_n \end{bmatrix}$$

Matrix Representation of a Linear System

Matrix Representation

If $\boldsymbol{A}$ is the coefficient matrix of a system of linear equations and
$\boldsymbol{b}$ is the vector of constants, then we will write $LS(\boldsymbol{A}, \boldsymbol{b})$
as a shorthand expression for the system of linear equations,
which we will refer to as the matrix representation of the linear system.

Augmented Matrix

Augmented Matrix

Suppose we have a system of m equations in n variables,
with coefficient matrix A and vector of constants b.

Then the augmented matrix of the system of equations is

the $m \times (n+1)$ matrix whose first n columns are the columns of A
and whose last column $(n+1)$ is the column vector b.
When described symbolically, this matrix will be written as $[\boldsymbol{A}|\boldsymbol{b}]$.

Augmented Matrix Example

Augmented Matrix Example

Archetype A is the following system of 3 equations in 3 variables.

$$\begin{array}{llll} x_1 & -x_2 & +2x_3 & = 1 \\ 2x_1 & +x_2 & +x_3 & = 8 \\ x_1 & +x_2 & & = 5 \end{array}$$

Here is its augmented matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{bmatrix}$$

Row Operations

Augmented Matrix Example

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a row operation.

1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j
2. αR_i: Multiply row i by the nonzero scalar α
3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j

Row Equivalent Matrices

Row Equivalent Matrices

Two matrices, A and B, are row-equivalent
if one can be obtained from the other by a sequence of row operations.

Two Row Equivalent Matrices

Row Equivalent Matrix Example

The matrices

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & 2 & -2 & 3 \\ 1 & 1 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 6 \\ 3 & 0 & -2 & -9 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

are row equivalent

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & 2 & -2 & 3 \\ 1 & 1 & 0 & 6 \end{bmatrix} : R_1 \leftrightarrow R_3 \quad \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 6 \\ 5 & 2 & -2 & 3 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 6 \\ 5 & 2 & -2 & 3 \\ 2 & -1 & 3 & 4 \end{bmatrix} : -2R_1 + R_2 \quad \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 6 \\ 3 & 0 & -2 & -9 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

Row Equivalent Matrices

Equivalent Systems

Suppose that A and B are row-equivalent augmented matrices.

Then the two systems of linear equations
represented by A and B are equivalent systems

Reduced Row-Echelon Form (1)

Reduced Row-Echelon Form

A matrix is in reduced row-echelon form
if it meets all of the following conditions:

1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
2. The leftmost nonzero entry of a row is equal to 1.
3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s>i, then t>j

Reduced Row-Echelon Form (2)

a **zero row**, a **leading 1**, a **pivot column**

A row of only zero entries is called a **zero row** and
the leftmost nonzero entry of a nonzero row is a **leading 1**.
A column containing a leading 1 will be called a **pivot column**.

The number of nonzero rows will be denoted by r,
which is also equal to the number of **leading 1**'s
and the number of **pivot columns**.

The set of column indices for the **pivot columns**
will be denoted by $D = d_1, d_2, d_3, \ldots, d_r$ where $d_1 < d_2 < d_3 < \cdots < d_r$,

while the columns that are not **pivot columns**
will be denoted as $F = f_1, f_2, f_3, \ldots, f_{n-r}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

A Reduced Row Echelon Matrix Example

A Reduced Row Echelon Matrix Example

The matrix C is in reduced row-echelon form.

$$C = \begin{bmatrix} 1 & -3 & 0 & 6 & 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix has two zero rows and three pivot columns.
So $r = 3$. Columns 1, 5, and 6 are the three pivot columns,
so $D = 1, 5, 6$ and then $F = 2, 3, 4, 7, 8$.

A Non Reduced Row Echelon Matrix Example

A Non Reduced Row Echelon Matrix Example

The matrix E is not in reduced row-echelon form, as it fails each of the four requirements once.

$$C = \begin{bmatrix} 1 & 0 & -3 & 0 & 6 & 0 & 7 & -5 & 9 \\ 0 & 0 & 0 & 5 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row-Equivalent Matrix in Echelon Form (1)

Row-Equivalent Matrix in Echelon Form

Suppose A is a matrix. Then there is a matrix B
so that

1. A and B are row-equivalent.
2. B is in reduced row-echelon form.

Suppose that A has m rows and n columns. We will describe a process for converting A into B via row operations. This procedure is known as Gauss-Jordan elimination. Tracing through this procedure will be easier if you recognize that i refers to a row that is being converted, j refers to a column that is being converted, and r keeps track of the number of nonzero rows. Here we go.

Row-Equivalent Matrix in Echelon Form (2)

Row-Equivalent Matrix in Echelon Form

Suppose that A has m rows and n columns. We will describe a process for converting A into B via row operations. This procedure is known as Gauss-Jordan elimination. Tracing through this procedure will be easier if you recognize that i refers to a row that is being converted, j refers to a column that is being converted, and r keeps track of the number of nonzero rows. Here we go.

Row-Equivalent Matrix in Echelon Form (3)

Row-Equivalent Matrix in Echelon Form

1. Set j=0 and r=0.
2. Increase j by 1. If j now equals n+1, then stop.
3. Examine the entries of A in column j located in rows r+1 through m. If all of these entries are zero, then go to Step 2.
4. Choose a row from rows r+1 through m with a nonzero entry in column j. Let i denote the index for this row.
   Increase r by 1.
5. Use the first row operation to swap rows i and r.
6. Use the second row operation to convert the entry in row r and column j to a 1.
7. Use the third row operation with row r to convert every other entry of column j to zero.
8. Go to Step 2.

Row-Equivalent Matrix in Echelon Form (4)

Row-Equivalent Matrix in Echelon Form

The result of this procedure is that the matrix A is converted to a matrix in reduced row-echelon form, which we will refer to as B. We need to now prove this claim by showing that the converted matrix has the requisite properties of Definition RREF. First, the matrix is only converted through row operations (Steps 6, 7, 8), so A and B

are row-equivalent (Definition REM).
It is a bit more work to be certain that B is in reduced row-echelon form. We claim that as we begin Step 2, the first j columns of the matrix are in reduced row-echelon form with r nonzero rows. Certainly this is true at the start when j=0, since the matrix has no columns and so vacuously meets the conditions of Definition RREF with r=0 nonzero rows.